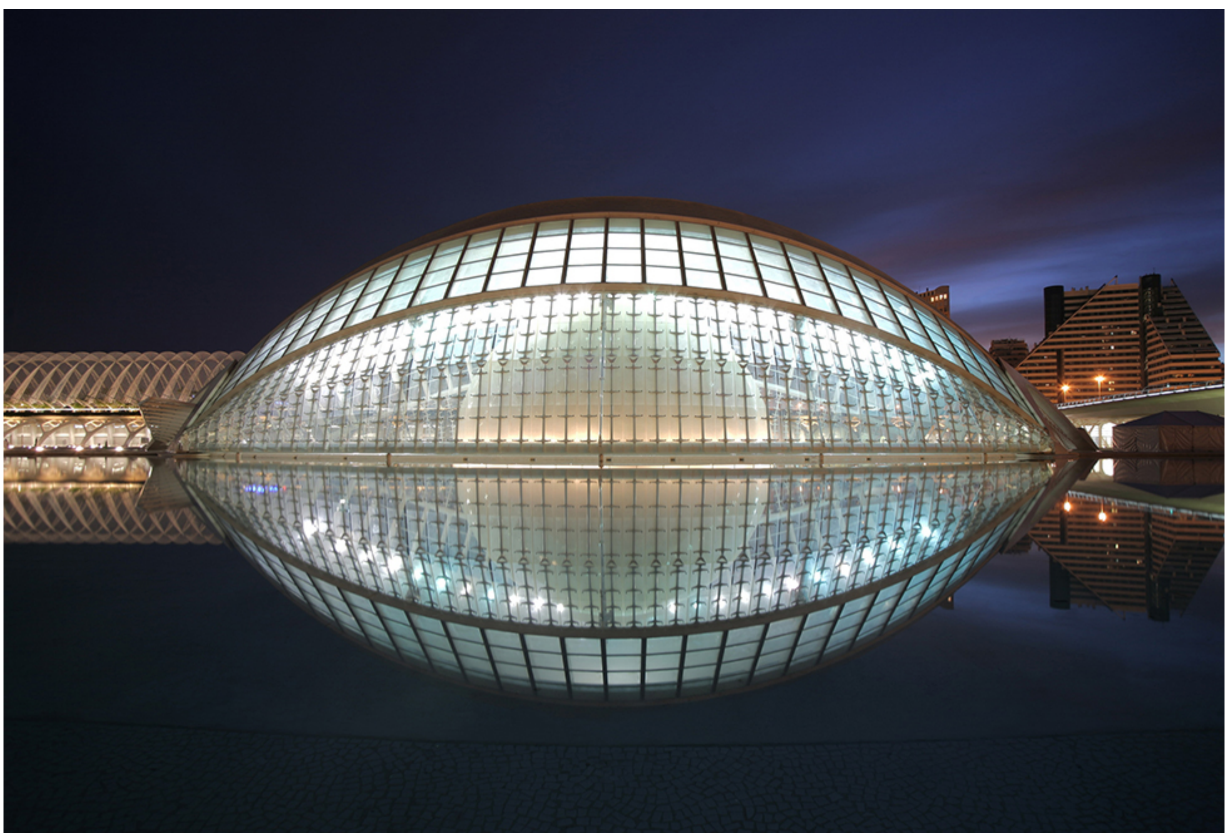
Introduction
class="introduction"

The City of Arts and Sciences in Valencia, Spain, has a unique structure along an axis of just two kilometers that was formerly the bed of the River Turia. The l’Hemisfèric has an IMAX cinema with three systems of modern digital projections onto a concave screen of 900 square meters. An oval roof over 100 meters long has been

made to look like a huge human eye that comes alive and opens up to the world as the “Eye of Wisdom.” (credit: modification of work by Javier Yaya Tur, Wikimedia Commons)



In this chapter we extend the concept of a definite integral of a single variable to double and triple integrals of functions of two and three variables, respectively. We examine applications involving integration to compute volumes, masses, and centroids of more general regions. We will also see how the use of other coordinate systems (such as polar, cylindrical, and spherical coordinates) makes it simpler to compute multiple integrals over some types of regions and functions. As an example, we will use polar coordinates to find the volume of structures such as l'Hemisfèric. (See [link].)

In the preceding chapter, we discussed differential calculus with multiple independent variables. Now we examine integral calculus in multiple dimensions. Just as a partial derivative allows us to differentiate a function with respect to one variable while holding the other variables constant, we will see that an iterated integral allows us to integrate a function with respect to one variable while holding the other variables constant.

Double Integrals over Rectangular Regions

- Recognize when a function of two variables is integrable over a rectangular region.
- Recognize and use some of the properties of double integrals.
- Evaluate a double integral over a rectangular region by writing it as an iterated integral.
- Use a double integral to calculate the area of a region, volume under a surface, or average value of a function over a plane region.

In this section we investigate double integrals and show how we can use them to find the volume of a solid over a rectangular region in the xy-plane. Many of the properties of double integrals are similar to those we have already discussed for single integrals.

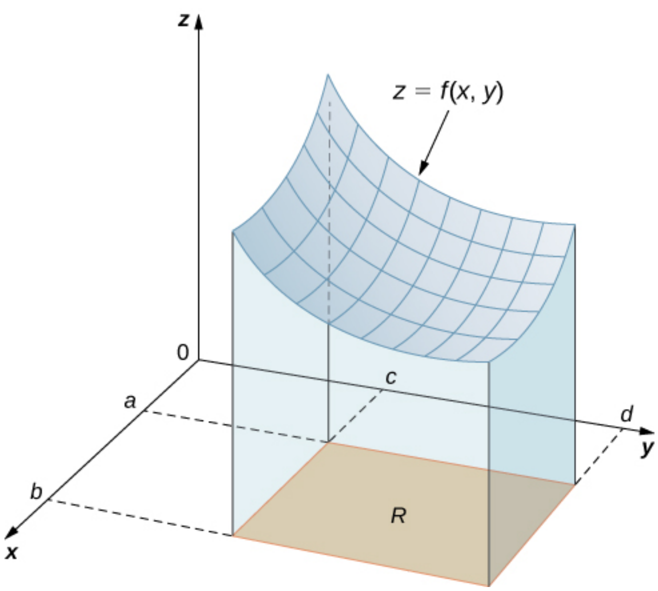
Volumes and Double Integrals

We begin by considering the space above a rectangular region R. Consider a continuous function $f\left(x,y\right)\ge 0$ of two variables defined on the closed rectangle R:

Equation:

$$R=\left[a,b\right]\,\times\,\left[c,d\right]=\left\{\left(x,y\right)\in \mathbb{R}^2|a\le x\le b,c\le y\le d\right\}$$

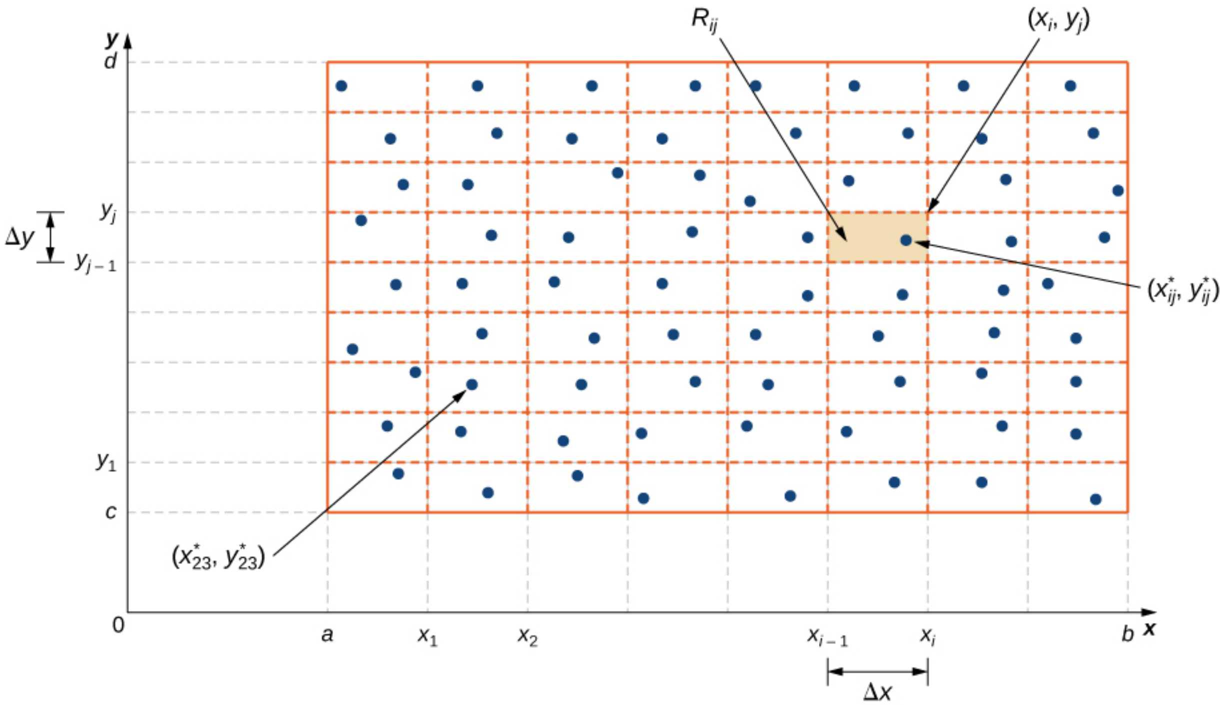
Here $\left[a,b\right]\,\times\,\left[c,d\right]$ denotes the Cartesian product of the two closed intervals $\left[a,b\right]$ and $\left[c,d\right]$. It consists of rectangular pairs $\left(x,y\right)$ such that $a\le x\le b$ and $c\le y\le d$. The graph of f represents a surface above the xy-plane with equation $z=f\left(x,y\right)$ where z is the height of the surface at the point $\left(x,y\right)$. Let S be the solid that lies above R and under the graph of f ([link]). The base of the solid is the rectangle R in the xy-plane. We want to find the volume V of the solid S.



The graph of $f\left(x,y\right)$ over the rectangle R in the xy-plane is a curved surface.

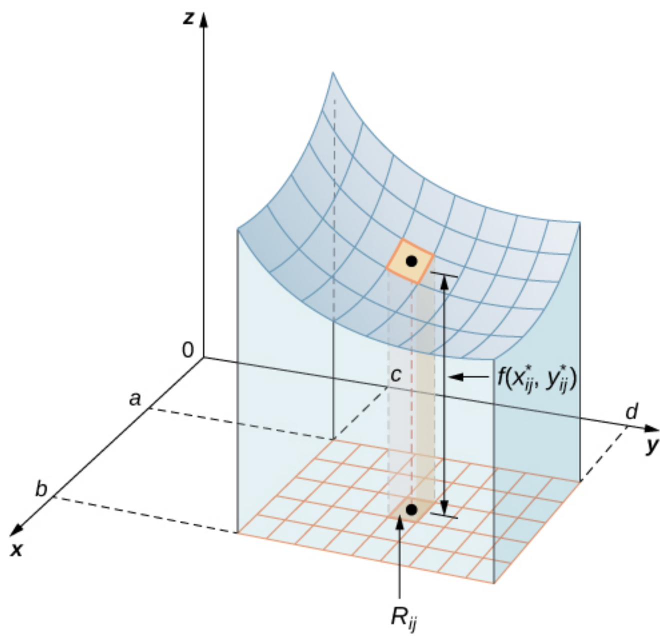
We divide the region R into small rectangles R_{ij}, each with area ΔA and with sides Δx and Δy ([link]). We do this by dividing the interval $\left[a,b\right]$ into m subintervals and dividing the interval $\left[c,d\right]$ into n subintervals. Hence

$\Delta x = \frac{b-a}{m}$, $\Delta y = \frac{d-c}{n}$, and $\Delta A = \Delta x \Delta y$.



Rectangle R is divided into small rectangles R_{ij}, each with area ΔA.

The volume of a thin rectangular box above R_{ij} is $f(x^*_{ij}, y^*_{ij})\Delta A$, where (x^*_{ij}, y^*_{ij}) is an arbitrary sample point in each R_{ij} as shown in the following figure.



A thin rectangular box above R_{ij} with height $f\left(x_{ij}^*, y_{ij}^*\right)$.

Using the same idea for all the subrectangles, we obtain an approximate volume of the solid S as $V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$. This sum is known as a **double Riemann sum** and can be used to approximate the value of the volume of the solid. Here the double sum means that for each subrectangle we evaluate the function at the chosen point, multiply by the area of each rectangle, and then add all the results.

As we have seen in the single-variable case, we obtain a better approximation to the actual volume if *m* and *n* become larger.

Equation:

$$V = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A \text{ or } V = \lim_{\Delta x, \Delta y\to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A.$$

Note that the sum approaches a limit in either case and the limit is the volume of the solid with the base *R*. Now we are ready to define the double integral.

Note:

Definition

The **double integral** of the function $f(x, y)$ over the rectangular region R in the xy-plane is defined as

Equation:

$$\iint_R f(x,y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A.$$

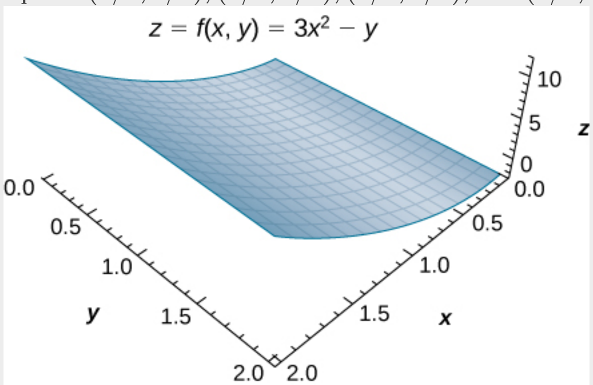
If $f(x, y) \geq 0$, then the volume V of the solid S, which lies above R in the xy-plane and under the graph of f, is the double integral of the function $f(x, y)$ over the rectangle R. If the function is ever negative, then the double integral can be considered a "signed" volume in a manner similar to the way we defined net signed area in The Definite Integral.

Example:
Exercise:

Problem:
Setting up a Double Integral and Approximating It by Double Sums

Consider the function $z = f(x, y) = 3x^2 - y$ over the rectangular region $R = [0, 2] \times [0, 2]$ ([link]).

a. Set up a double integral for finding the value of the signed volume of the solid S that lies above R and "under" the graph of f.
b. Divide R into four squares with $m = n = 2$, and choose the sample point as the upper right corner point of each square $(1, 1)$, $(2, 1)$, $(1, 2)$, and $(2, 2)$ ([link]) to approximate the signed volume of the solid S that lies above R and "under" the graph of f.
c. Divide R into four squares with $m = n = 2$, and choose the sample point as the midpoint of each square: $(1/2, 1/2)$, $(3/2, 1/2)$, $(1/2, 3/2)$, and $(3/2, 3/2)$ to approximate the signed volume.



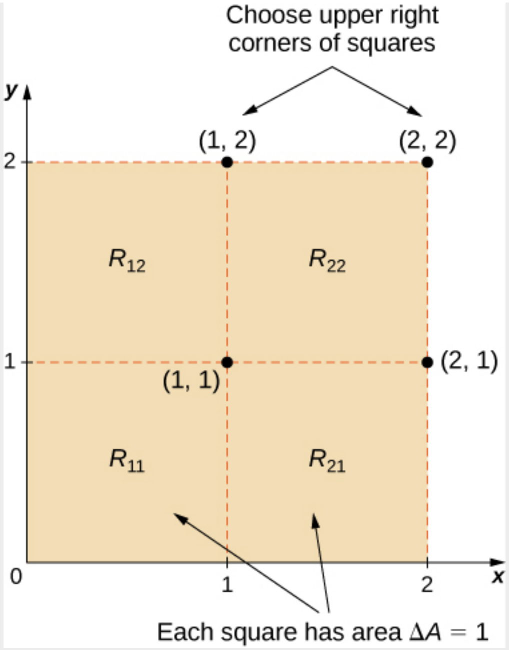
The function $z = f(x, y)$ graphed over the rectangular region $R = [0, 2] \times [0, 2]$.

Solution:

a. As we can see, the function $z = f(x, y) = 3x^2 - y$ is above the plane. To find the signed volume of S, we need to divide the region R into small rectangles R_{ij}, each with area ΔA and with sides Δx and Δy, and choose (x^*_{ij}, y^*_{ij}) as sample points in each R_{ij}. Hence, a double integral is set up as
Equation:

$$V = \iint_R (3x^2 - y)\, dA = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \left[3\left(x^*_{ij}\right)^2 - y^*_{ij} \right] \Delta A.$$

b. Approximating the signed volume using a Riemann sum with $m = n = 2$ we have $\Delta A = \Delta x \Delta y = 1 \times 1 = 1$. Also, the sample points are (1, 1), (2, 1), (1, 2), and (2, 2) as shown in the following figure.



Subrectangles for the rectangular region $R = [0, 2] \times [0, 2]$.

Hence,
Equation:

$$\begin{aligned} V &= \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_{ij}^{*}, y_{ij}^{*})\Delta A \\ &= \sum_{i=1}^{2}(f(x_{i1}^{*}, y_{i1}^{*}) + f(x_{i2}^{*}, y_{i2}^{*}))\Delta A \\ &= f(x_{11}^{*}, y_{11}^{*})\Delta A + f(x_{21}^{*}, y_{21}^{*})\Delta A + f(x_{12}^{*}, y_{12}^{*})\Delta A + f(x_{22}^{*}, y_{22}^{*})\Delta A \\ &= f(1,1)(1) + f(2,1)(1) + f(1,2)(1) + f(2,2)(1) \\ &= (3-1)(1) + (12-1)(1) + (3-2)(1) + (12-2)(1) \\ &= 2 + 11 + 1 + 10 = 24. \end{aligned}$$

c. Approximating the signed volume using a Riemann sum with $m = n = 2$, we have $\Delta A = \Delta x \Delta y = 1 \times 1 = 1$. In this case the sample points are (1/2, 1/2), (3/2, 1/2), (1/2, 3/2), and (3/2, 3/2).
Hence
Equation:

$$\begin{aligned} V &= \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_{ij}^{*}, y_{ij}^{*})\Delta A \\ &= f(x_{11}^{*}, y_{11}^{*})\Delta A + f(x_{21}^{*}, y_{21}^{*})\Delta A + f(x_{12}^{*}, y_{12}^{*})\Delta A + f(x_{22}^{*}, y_{22}^{*})\Delta A \\ &= f(1/2, 1/2)(1) + f(3/2, 1/2)(1) + f(1/2, 3/2)(1) + f(3/2, 3/2)(1) \\ &= (\tfrac{3}{4} - \tfrac{1}{4})(1) + (\tfrac{27}{4} - \tfrac{1}{2})(1) + (\tfrac{3}{4} - \tfrac{3}{2})(1) + (\tfrac{27}{4} - \tfrac{3}{2})(1) \\ &= \tfrac{2}{4} + \tfrac{25}{4} + \left(-\frac{3}{4}\right) + \tfrac{21}{4} = \tfrac{45}{4} = 11. \end{aligned}$$

Analysis

Notice that the approximate answers differ due to the choices of the sample points. In either case, we are introducing some error because we are using only a few sample points. Thus, we need to investigate how we can achieve an accurate answer.

Note:
Exercise:

Problem: Use the same function $z = f(x, y) = 3x^2 - y$ over the rectangular region $R = [0, 2] \times [0, 2]$.

Divide R into the same four squares with $m = n = 2$, and choose the sample points as the upper left corner point of each square $(0, 1), (1, 1), (0, 2)$, and $(1, 2)$ ([link]) to approximate the signed volume of the solid S that lies above R and "under" the graph of f.

Solution:

$$V = \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_{ij}^{*}, y_{ij}^{*})\Delta A = 0$$

Hint

Follow the steps of the previous example.

Note that we developed the concept of double integral using a rectangular region R. This concept can be extended to any general region. However, when a region is not rectangular, the subrectangles may not all fit perfectly into R, particularly if the base area is curved. We examine this situation in more detail in the next section, where we study regions that are not always rectangular and subrectangles may not fit perfectly in the region R. Also, the heights may not be exact if the surface $z = f(x, y)$ is curved. However, the errors on the sides and the height where the pieces may not fit perfectly within the solid S approach 0 as m and n approach infinity. Also, the double integral of the function $z = f(x, y)$ exists provided that the function f is not too discontinuous. If the function is bounded and continuous over R except on a finite number of smooth curves, then the double integral exists and we say that f is integrable over R.

Since $\Delta A = \Delta x \Delta y = \Delta y \Delta x$, we can express dA as $dx\,dy$ or $dy\,dx$. This means that, when we are using rectangular coordinates, the double integral over a region R denoted by $\iint\limits_R f(x, y)dA$ can be written as $\iint\limits_R f(x, y)dx\,dy$ or $\iint\limits_R f(x, y)dy\,dx$.

Now let's list some of the properties that can be helpful to compute double integrals.

Properties of Double Integrals

The properties of double integrals are very helpful when computing them or otherwise working with them. We list here six properties of double integrals. Properties 1 and 2 are referred to as the linearity of the integral, property 3 is the additivity of the integral, property 4 is the monotonicity of the integral, and property 5 is used to find the bounds of the integral. Property 6 is used if $f(x,y)$ is a product of two functions $g(x)$ and $h(y)$.

Note:
Properties of Double Integrals
Assume that the functions $f(x,y)$ and $g(x,y)$ are integrable over the rectangular region *R*; *S* and *T* are subregions of *R*; and assume that *m* and *M* are real numbers.

i. The sum $f(x,y)+g(x,y)$ is integrable and
Equation:

$$\iint\limits_R [f(x,y)+g(x,y)]dA = \iint\limits_R f(x,y)dA + \iint\limits_R g(x,y)dA.$$

ii. If *c* is a constant, then $cf(x,y)$ is integrable and
Equation:

$$\iint\limits_R cf(x,y)dA = c\iint\limits_R f(x,y)dA.$$

iii. If $R = S \cup T$ and $S \cap T = \emptyset$ except an overlap on the boundaries, then
Equation:

$$\iint\limits_R f(x,y)dA = \iint\limits_S f(x,y)dA + \iint\limits_T f(x,y)dA.$$

iv. If $f(x,y) \geq g(x,y)$ for (x,y) in R, then
Equation:

$$\iint\limits_R f(x,y)dA \geq \iint\limits_R g(x,y)dA.$$

v. If $m \leq f(x,y) \leq M$, then
Equation:

$$m \times A(R) \leq \iint\limits_R f(x,y)dA \leq M \times A(R).$$

vi. In the case where $f(x,y)$ can be factored as a product of a function $g(x)$ of x only and a function $h(y)$ of y only, then over the region $R = \{(x,y) | a \leq x \leq b, c \leq y \leq d\}$, the double integral can be written as
Equation:

$$\iint\limits_R f(x,y)dA = \left(\int_a^b g(x)dx\right)\left(\int_c^d h(y)dy\right).$$

These properties are used in the evaluation of double integrals, as we will see later. We will become skilled in using these properties once we become familiar with the computational tools of double integrals. So let's get to that now.

Iterated Integrals

So far, we have seen how to set up a double integral and how to obtain an approximate value for it. We can also imagine that evaluating double integrals by using the definition can be a very lengthy process if we choose larger values for m and n. Therefore, we need a practical and convenient technique for computing double integrals. In other words, we need to learn how to compute double integrals without employing the definition that uses limits and double sums.

The basic idea is that the evaluation becomes easier if we can break a double integral into single integrals by integrating first with respect to one variable and then with respect to the other. The key tool we need is called an iterated integral.

Note:
Definition
Assume $a, b, c,$ and d are real numbers. We define an **iterated integral** for a function $f(x, y)$ over the rectangular region $R = [a, b] \times [c, d]$ as

a. **Equation:**

$$\int_a^b \int_c^d f(x, y) dy\, dx = \int_a^b \left[\int_c^d f(x, y) dy \right] dx$$

b. **Equation:**

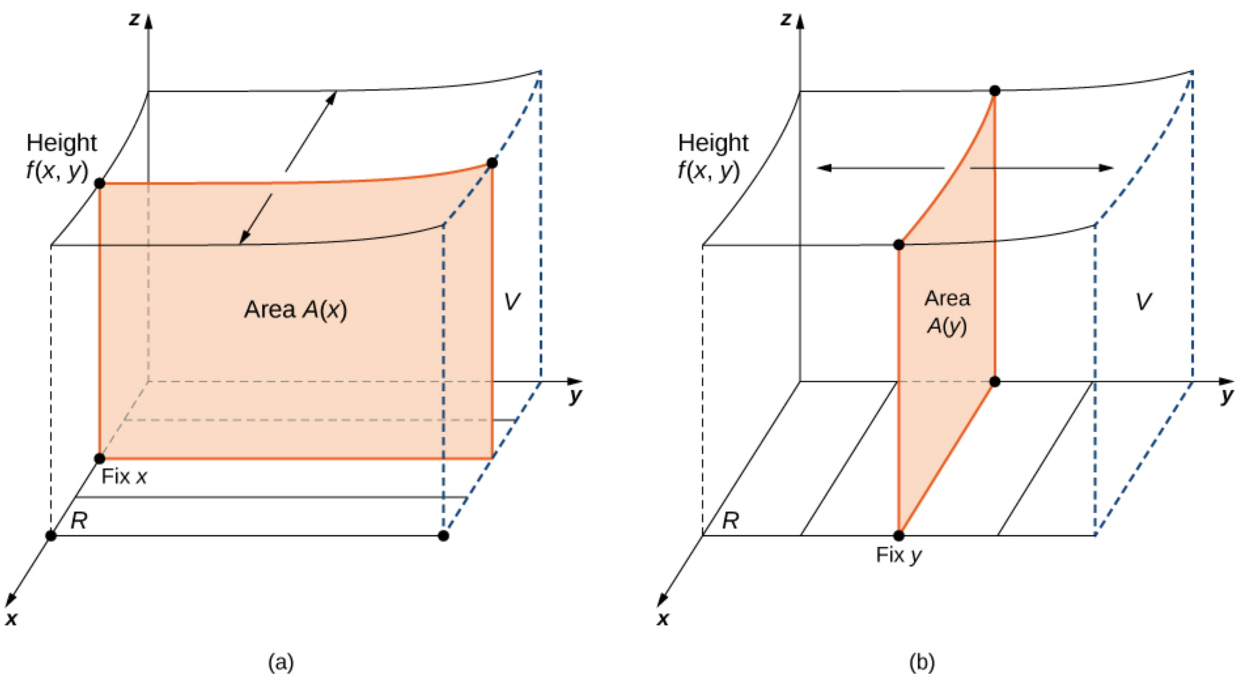
$$\int_c^d \int_a^b f(x, y) dx\, dy = \int_c^d \left[\int_a^b f(x, y) dx \right] dy.$$

The notation $\int_a^b \left[\int_c^d f(x, y) dy \right] dx$ means that we integrate $f(x, y)$ with respect to *y* while holding *x* constant. Similarly, the notation $\int_c^d \left[\int_a^b f(x, y) dx \right] dy$ means that we integrate $f(x, y)$ with respect to *x* while holding *y* constant. The fact that double integrals can be split into iterated integrals is expressed in Fubini's theorem. Think of this theorem as an essential tool for evaluating double integrals.

Note:
Fubini's Theorem
Suppose that $f(x, y)$ is a function of two variables that is continuous over a rectangular region $R = \{(x, y) \in \mathbb{R}^2 | a \le x \le b, \mathrm{c} \le \mathrm{y} \le \mathrm{d}\}$. Then we see from [link] that the double integral of f over the region equals an iterated integral,
Equation:

$$\iint\limits_R f(x, y) dA = \iint\limits_R f(x, y) dx\, dy = \int\limits_a^b \int\limits_c^d f(x, y) dy\, dx = \int\limits_c^d \int\limits_a^b f(x, y) dx\, dy.$$

More generally, **Fubini's theorem** is true if f is bounded on R and f is discontinuous only on a finite number of continuous curves. In other words, f has to be integrable over R.



(a) Integrating first with respect to y and then with respect to x to find the area $A(x)$ and then the volume V; (b) integrating first with respect to x and then with respect to y to find the area $A(y)$ and then the volume V.

Example:
Exercise:

Problem:
Using Fubini's Theorem

Use Fubini's theorem to compute the double integral $\iint\limits_R f(x, y) dA$ where $f(x, y) = x$ and $R = [0, 2] \times [0, 1]$.

Solution:

Fubini's theorem offers an easier way to evaluate the double integral by the use of an iterated integral. Note how the boundary values of the region *R* become the upper and lower limits of integration.

Equation:

$$\begin{aligned}\iint_R f(x,y)dA &= \iint_R f(x,y)dx\,dy \\ &= \int_{y=0}^{y=1}\int_{x=0}^{x=2} x\,dx\,dy \\ &= \int_{y=0}^{y=1}\left[\frac{x^2}{2}\Big|_{x=0}^{x=2}\right]dy \\ &= \int_{y=0}^{y=1} 2dy = 2y\Big|_{y=0}^{y=1} = 2.\end{aligned}$$

The double integration in this example is simple enough to use Fubini's theorem directly, allowing us to convert a double integral into an iterated integral. Consequently, we are now ready to convert all double integrals to iterated integrals and demonstrate how the properties listed earlier can help us evaluate double integrals when the function $f(x,y)$ is more complex. Note that the order of integration can be changed (see [link]).

Example:
Exercise:

Problem:
Illustrating Properties i and ii

Evaluate the double integral $\iint_R (xy - 3xy^2)dA$ where $R = \{(x,y)|0 \le x \le 2, 1 \le y \le 2\}$.

Solution:

This function has two pieces: one piece is xy and the other is $3xy^2$. Also, the second piece has a constant 3. Notice how we use properties i and ii to help evaluate the double integral.

Equation:

$$\iint_R \left(xy - 3xy^2\right) dA$$

$$= \iint_R xy\, dA + \iint_R \left(-3xy^2\right) dA$$ Property i: Integral of a sum is the sum of the int

$$= \int_{y=1}^{y=2} \int_{x=0}^{x=2} xy\, dx\, dy - \int_{y=1}^{y=2} \int_{x=0}^{x=2} 3xy^2 dx\, dy$$ Convert double integrals to iterated integrals.

$$= \int_{y=1}^{y=2} \left(\tfrac{x^2}{2} y\right)\Big|_{x=0}^{x=2} dy - 3 \int_{y=1}^{y=2} \left(\tfrac{x^2}{2} y^2\right)\Big|_{x=0}^{x=2} dy$$ Integrate with respect to x, holding y constant.

$$= \int_{y=1}^{y=2} 2y\, dy - \int_{y=1}^{y=2} 6y^2 dy$$ Property ii: Placing the constant before the integ

$$= \int_1^2 y\, dy - 6 \int_1^2 y^2 dy$$ Integrate with respect to y.

$$= 2\tfrac{y^2}{2}\Big|_1^2 - 6\tfrac{y^3}{3}\Big|_1^2$$

$$= y^2\Big|_1^2 - 2y^3\Big|_1^2$$

$$= (4-1) - 2(8-1)$$

$$= 3 - 2(7) = 3 - 14 = -11.$$

Example:
Exercise:

Problem:
Illustrating Property v.

Over the region $R = \{(x, y) | 1 \leq x \leq 3, 1 \leq y \leq 2\}$, we have $2 \leq x^2 + y^2 \leq 13$. Find a lower and an upper bound for the integral $\iint_R \left(x^2 + y^2\right) dA$.

Solution:

For a lower bound, integrate the constant function 2 over the region R. For an upper bound, integrate the constant function 13 over the region R.
Equation:

$$\begin{aligned} \int_1^2 \int_1^3 2dx\, dy &= \int_1^2 \left[2x\big|_1^3\right] dy = \int_1^2 2(2) dy = 4y\big|_1^2 = 4(2-1) = 4 \\ \int_1^2 \int_1^3 13dx\, dy &= \int_1^2 \left[13x\big|_1^3\right] dy = \int_1^2 13(2) dy = 26y\big|_1^2 = 26(2-1) = 26. \end{aligned}$$

Hence, we obtain $4 \leq \iint_R \left(x^2 + y^2\right) dA \leq 26.$

Example:
Exercise:

Problem:
Illustrating Property vi

Evaluate the integral $\iint\limits_R e^y \cos x \, dA$ over the region $R = \left\{(x, y) \middle| 0 \le x \le \frac{\pi}{2}, 0 \le y \le 1\right\}$.

Solution:

This is a great example for property vi because the function $f(x, y)$ is clearly the product of two single-variable functions e^y and $\cos x$. Thus we can split the integral into two parts and then integrate each one as a single-variable integration problem.
Equation:

$$\begin{aligned} \iint\limits_R e^y \cos x \, dA &= \int_0^1 \int_0^{\pi/2} e^y \cos x \, dx \, dy \\ &= \left(\int_0^1 e^y dy\right) \left(\int_0^{\pi/2} \cos x \, dx\right) \\ &= \left(e^y \Big|_0^1\right) \left(\sin x \Big|_0^{\pi/2}\right) \\ &= e - 1. \end{aligned}$$

Note:
Exercise:

Problem:

a. Use the properties of the double integral and Fubini's theorem to evaluate the integral
Equation:

$$\int_0^1 \int_{-1}^3 (3 - x + 4y) dy \, dx.$$

b. Show that $0 \le \iint\limits_R \sin \pi x \cos \pi y \, dA \le \dfrac{1}{32}$ where $R = \left(0, \frac{1}{4}\right)\left(\frac{1}{4}, \frac{1}{2}\right)$.

Solution:

a. 26 b. Answers may vary.

Hint

Use properties i. and ii. and evaluate the iterated integral, and then use property v.

As we mentioned before, when we are using rectangular coordinates, the double integral over a region R denoted by $\iint\limits_R f(x,y)dA$ can be written as $\iint\limits_R f(x,y)dx\,dy$ or $\iint\limits_R f(x,y)dy\,dx$. The next example shows that the results are the same regardless of which order of integration we choose.

Example:
Exercise:

Problem:
Evaluating an Iterated Integral in Two Ways

Let's return to the function $f(x,y) = 3x^2 - y$ from [link], this time over the rectangular region $R = [0,2] \times [0,3]$. Use Fubini's theorem to evaluate $\iint\limits_R f(x,y)dA$ in two different ways:

a. First integrate with respect to *y* and then with respect to *x*;
b. First integrate with respect to *x* and then with respect to *y*.

Solution:

[link] shows how the calculation works in two different ways.

a. First integrate with respect to *y* and then integrate with respect to *x*:
Equation:

$$\begin{aligned}\iint\limits_R f(x,y)dA &= \int_{x=0}^{x=2}\int_{y=0}^{y=3}(3x^2-y)dy\,dx \\ &= \int_{x=0}^{x=2}\left(\int\limits_{y=0}^{y=3}(3x^2-y)dy\right)dx = \int_{x=0}^{x=2}\left[3x^2y - \frac{y^2}{2}\Big|_{y=0}^{y=3}\right]dx \\ &= \int_{x=0}^{x=2}(9x^2-\tfrac{9}{2})dx = 3x^3 - \tfrac{9}{2}x\Big|_{x=0}^{x=2} = 15.\end{aligned}$$

b. First integrate with respect to *x* and then integrate with respect to *y*:
Equation:

$$\begin{aligned}\iint\limits_R f(x,y)dA &= \int_{y=0}^{y=3}\int_{x=0}^{x=2}(3x^2-y)dx\,dy \\ &= \int_{y=0}^{y=3}\left(\int_{x=0}^{x=2}(3x^2-y)dx\right)dy = \int_{y=0}^{y=3}\left[x^3 - xy\Big|_{x=0}^{x=2}\right]dy \\ &= \int_{y=0}^{y=3}(8-2y)dy = 8y - y^2\Big|_{y=0}^{y=3} = 15.\end{aligned}$$

Analysis

With either order of integration, the double integral gives us an answer of 15. We might wish to interpret this answer as a volume in cubic units of the solid S below the function $f(x,y) = 3x^2 - y$ over the region

$R = [0, 2] \times [0, 3]$. However, remember that the interpretation of a double integral as a (non-signed) volume works only when the integrand f is a nonnegative function over the base region R.

Note:
Exercise:

Problem: Evaluate $\int_{y=-3}^{y=2} \int_{x=3}^{x=5} (2 - 3x^2 + y^2) dx\, dy$.

Solution:

$-\frac{1340}{3}$

Hint

Use Fubini's theorem.

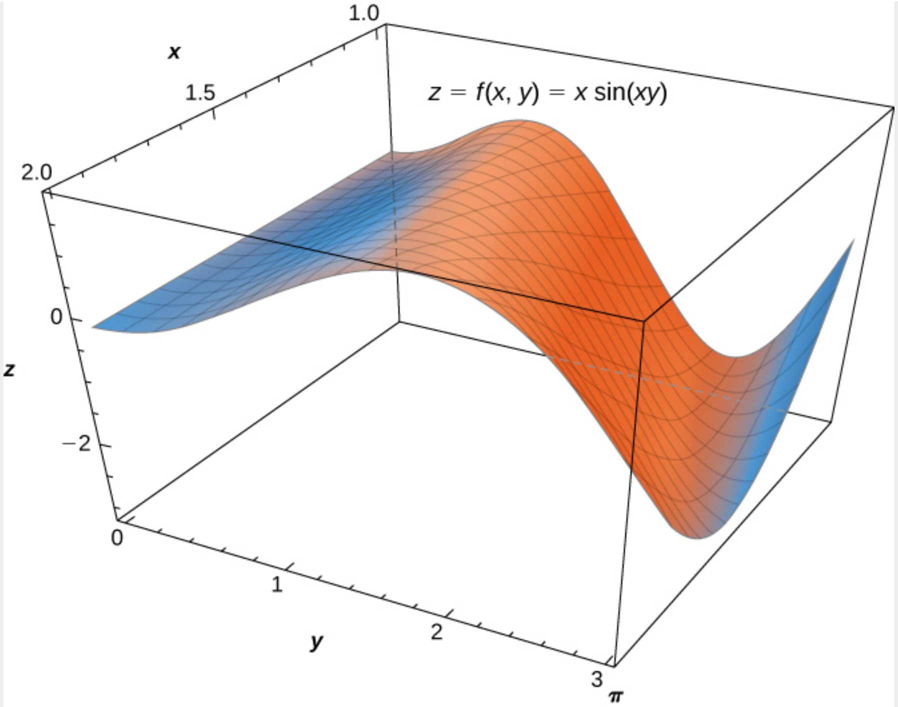
In the next example we see that it can actually be beneficial to switch the order of integration to make the computation easier. We will come back to this idea several times in this chapter.

Example:
Exercise:

Problem:
Switching the Order of Integration

Consider the double integral $\iint_R x \sin(xy) dA$ over the region $R = \{(x, y) | 0 \le x \le 3, 0 \le y \le 2\}$ ([link]).

a. Express the double integral in two different ways.
b. Analyze whether evaluating the double integral in one way is easier than the other and why.
c. Evaluate the integral.



The function $z = f(x, y) = x \sin(xy)$ over the rectangular region $R = [0, \pi] \times [1, 2]$.

Solution:

a. We can express $\iint\limits_R x \sin(xy) dA$ in the following two ways: first by integrating with respect to y and then with respect to x; second by integrating with respect to x and then with respect to y.
Equation:

$$\begin{aligned} &\iint\limits_R x \sin(xy) dA \\ &= \int_{x=0}^{x=\pi} \int_{y=1}^{y=2} x \sin(xy) dy\, dx && \text{Integrate first with respect to } y. \\ &= \int_{y=1}^{y=2} \int_{x=0}^{x=\pi} x \sin(xy) dx\, dy && \text{Integrate first with respect to } x. \end{aligned}$$

b. If we want to integrate with respect to y first and then integrate with respect to x, we see that we can use the substitution $u = xy$, which gives $du = x\, dy$. Hence the inner integral is simply $\int \sin u\, du$ and we can change the limits to be functions of x,
Equation:

$$\iint_R x\sin(xy)dA = \int_{x=0}^{x=\pi}\int_{y=1}^{y=2} x\sin(xy)dy\,dx = \int_{x=0}^{x=\pi}\left[\int_{u=x}^{u=2x}\sin(u)du\right]dx.$$

However, integrating with respect to x first and then integrating with respect to y requires integration by parts for the inner integral, with $u = x$ and $dv = \sin(xy)dx$.
Then $du = dx$ and $v = -\frac{\cos(xy)}{y}$, so
Equation:

$$\iint_R x\sin(xy)dA = \int_{y=1}^{y=2}\int_{x=0}^{x=\pi} x\sin(xy)dx\,dy = \int_{y=1}^{y=2}\left[-\frac{x\cos(xy)}{y}\Big|_{x=0}^{x=\pi} + \frac{1}{y}\int_{x=0}^{x=\pi}\cos(xy)dx\right]dy.$$

Since the evaluation is getting complicated, we will only do the computation that is easier to do, which is clearly the first method.

c. Evaluate the double integral using the easier way.
Equation:

$$\begin{aligned}\iint_R x\sin(xy)dA &= \int_{x=0}^{x=\pi}\int_{y=1}^{y=2} x\sin(xy)dy\,dx \\ &= \int_{x=0}^{x=\pi}\left[\int_{u=x}^{u=2x}\sin(u)du\right]dx = \int_{x=0}^{x=\pi}\left[-\cos u|_{u=x}^{u=2x}\right]dx = \int_{x=0}^{x=\pi}(-\cos 2x + \cos x)dx \\ &= -\tfrac{1}{2}\sin 2x + \sin x|_{x=0}^{x=\pi} = 0.\end{aligned}$$

Note:
Exercise:

Problem: Evaluate the integral $\iint_R xe^{xy}dA$ where $R = [0,1] \times [0,\ln 5]$.

Solution:

$\frac{4-\ln 5}{\ln 5}$

Hint

Integrate with respect to y first.

Applications of Double Integrals

Double integrals are very useful for finding the area of a region bounded by curves of functions. We describe this situation in more detail in the next section. However, if the region is a rectangular shape, we can find its area by

integrating the constant function $f(x,y)=1$ over the region R.

Note:
Definition
The area of the region R is given by $A(R)=\iint\limits_R 1dA$.

This definition makes sense because using $f(x,y)=1$ and evaluating the integral make it a product of length and width. Let's check this formula with an example and see how this works.

Example:
Exercise:

Problem:
Finding Area Using a Double Integral

Find the area of the region $R=\{(x,y)|0\le x\le 3, 0\le y\le 2\}$ by using a double integral, that is, by integrating 1 over the region R.

Solution:

The region is rectangular with length 3 and width 2, so we know that the area is 6. We get the same answer when we use a double integral:
Equation:

$$A(R)=\int_0^2\int_0^3 1dx\,dy=\int_0^2\left[x|_0^3\right]dy=\int_0^2 3dy=3\int_0^2 dy=3y|_0^2=3(2)=6.$$

We have already seen how double integrals can be used to find the volume of a solid bounded above by a function $f(x,y)$ over a region R provided $f(x,y)\ge 0$ for all (x,y) in R. Here is another example to illustrate this concept.

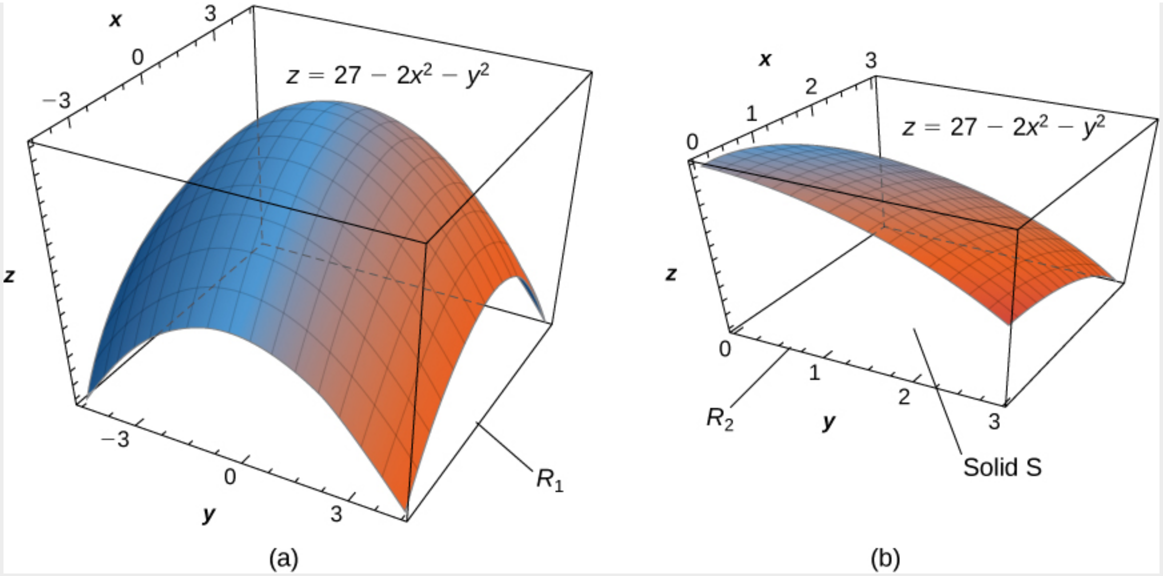
Example:
Exercise:

Problem:
Volume of an Elliptic Paraboloid

Find the volume V of the solid S that is bounded by the elliptic paraboloid $2x^2+y^2+z=27$, the planes $x=3$ and $y=3$, and the three coordinate planes.

Solution:

First notice the graph of the surface $z=27-2x^2-y^2$ in [link](a) and above the square region $R_1=[-3,3]\times[-3,3]$. However, we need the volume of the solid bounded by the elliptic paraboloid $2x^2+y^2+z=27$, the planes $x=3$ and $y=3$, and the three coordinate planes.



(a) The surface $z = 27 - 2x^2 - y^2$ above the square region $R_1 = [-3, 3] \times [-3, 3]$. (b) The solid S lies under the surface $z = 27 - 2x^2 - y^2$ above the square region $R_2 = [0, 3] \times [0, 3]$.

Now let's look at the graph of the surface in [link](b). We determine the volume V by evaluating the double integral over R_2:

Equation:

$$\begin{aligned} V &= \iint\limits_R z\,dA = \iint\limits_R \left(27 - 2x^2 - y^2\right) dA \\ &= \int_{y=0}^{y=3} \int_{x=0}^{x=3} \left(27 - 2x^2 - y^2\right) dx\,dy && \text{Convert to iterated integral.} \\ &= \int_{y=0}^{y=3} \left[27x - \tfrac{2}{3}x^3 - y^2 x\right]\Big|_{x=0}^{x=3} dy && \text{Integrate with respect to } x. \\ &= \int_{y=0}^{y=3} \left(64 - 3y^2\right) dy = 63y - y^3\Big|_{y=0}^{y=3} = 162. \end{aligned}$$

Note:
Exercise:

Problem:

Find the volume of the solid bounded above by the graph of $f(x, y) = xy\sin(x^2 y)$ and below by the xy-plane on the rectangular region $R = [0, 1] \times [0, \pi]$.

Solution:

$\frac{\pi}{2}$

Hint

Graph the function, set up the integral, and use an iterated integral.

Recall that we defined the average value of a function of one variable on an interval $[a, b]$ as

Equation:

$$f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x)dx.$$

Similarly, we can define the average value of a function of two variables over a region *R*. The main difference is that we divide by an area instead of the width of an interval.

Note:

Definition

The average value of a function of two variables over a region R is

Equation:

$$f_{\text{ave}} = \frac{1}{\text{Area } R} \iint_R f(x, y)dA.$$

In the next example we find the average value of a function over a rectangular region. This is a good example of obtaining useful information for an integration by making individual measurements over a grid, instead of trying to find an algebraic expression for a function.

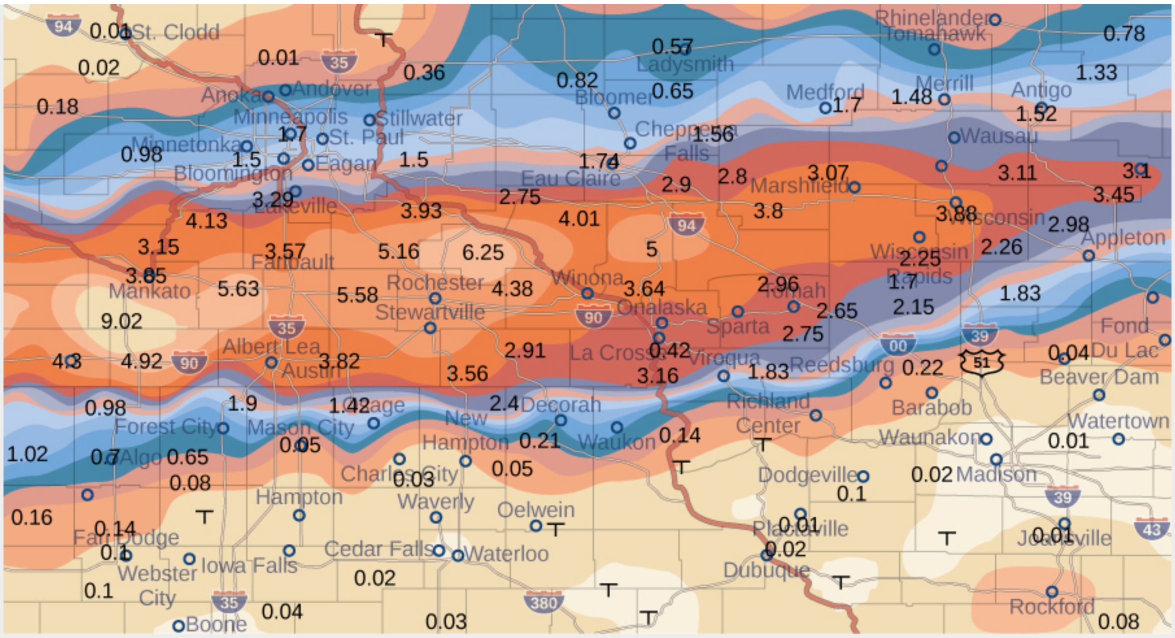
Example:

Exercise:

Problem:

Calculating Average Storm Rainfall

The weather map in [link] shows an unusually moist storm system associated with the remnants of Hurricane Karl, which dumped 4–8 inches (100–200 mm) of rain in some parts of the Midwest on September 22–23, 2010. The area of rainfall measured 300 miles east to west and 250 miles north to south. Estimate the average rainfall over the entire area in those two days.



Effects of Hurricane Karl, which dumped 4–8 inches (100–200 mm) of rain in some parts of southwest Wisconsin, southern Minnesota, and southeast South Dakota over a span of 300 miles east to west and 250 miles north to south.

Solution:

Place the origin at the southwest corner of the map so that all the values can be considered as being in the first quadrant and hence all are positive. Now divide the entire map into six rectangles $(m = 2 \text{ and } n = 3)$, as shown in [link]. Assume $f(x, y)$ denotes the storm rainfall in inches at a point approximately x miles to the east of the origin and y miles to the north of the origin. Let R represent the entire area of $250 \times 300 = 75000$ square miles. Then the area of each subrectangle is

Equation:

$$\Delta A = \frac{1}{6}(75000) = 12500.$$

Assume (x_{ij}^*, y_{ij}^*) are approximately the midpoints of each subrectangle R_{ij}. Note the color-coded region at each of these points, and estimate the rainfall. The rainfall at each of these points can be estimated as:

At (x_{11}, y_{11}) the rainfall is 0.08.

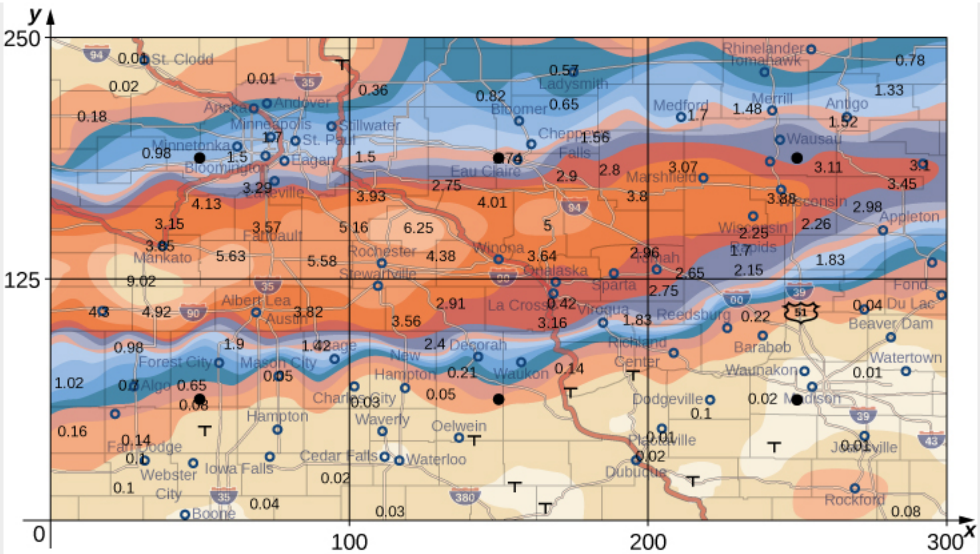
At (x_{12}, y_{12}) the rainfall is 0.08.

At (x_{13}, y_{13}) the rainfall is 0.01.

At (x_{21}, y_{21}) the rainfall is 1.70.

At (x_{22}, y_{22}) the rainfall is 1.74.

At (x_{23}, y_{23}) the rainfall is 3.00.



Storm rainfall with rectangular axes and showing the midpoints of each subrectangle.

According to our definition, the average storm rainfall in the entire area during those two days was

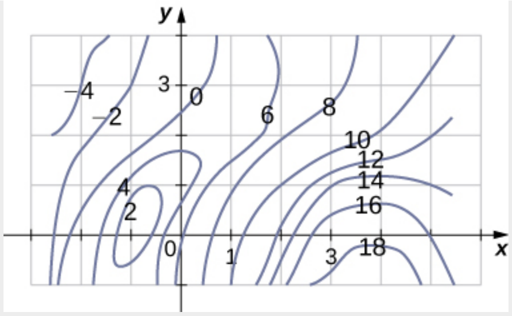
Equation:

$$\begin{aligned}
f_{\text{ave}} &= \frac{1}{\text{Area } R}\iint\limits_R f(x,y)\,dx\,dy = \frac{1}{75000}\iint\limits_R f(x,y)\,dx\,dy \\
&\cong \frac{1}{75{,}000}\sum_{i=1}^{3}\sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*)\Delta A \\
&\cong \tfrac{1}{75{,}000}\left[f(x_{11}^*, y_{11}^*)\Delta A + f(x_{12}^*, y_{12}^*)\Delta A\right. \\
&\quad \left.+ f(x_{13}^*, y_{13}^*)\Delta A + f(x_{21}^*, y_{21}^*)\Delta A + f(x_{22}^*, y_{22}^*)\Delta A + f(x_{23}^*, y_{23}^*)\Delta A\right] \\
&\cong \tfrac{1}{75{,}000}[0.08 + 0.08 + 0.01 + 1.70 + 1.74 + 3.00]\Delta A \\
&\cong \tfrac{1}{75{,}000}[0.08 + 0.08 + 0.01 + 1.70 + 1.74 + 3.00]12500 \\
&\cong \tfrac{5}{30}[0.08 + 0.08 + 0.01 + 1.70 + 1.74 + 3.00] \\
&\cong 1.10.
\end{aligned}$$

During September 22–23, 2010 this area had an average storm rainfall of approximately 1.10 inches.

Note:
Exercise:

Problem: A contour map is shown for a function $f(x, y)$ on the rectangle $R = [-3, 6] \times [-1, 4]$.



a. Use the midpoint rule with $m = 3$ and $n = 2$ to estimate the value of $\iint_R f(x, y)dA$.

b. Estimate the average value of the function $f(x, y)$.

Solution:

Answers to both parts a. and b. may vary.

Hint

Divide the region into six rectangles, and use the contour lines to estimate the values for $f(x, y)$.

Key Concepts

- We can use a double Riemann sum to approximate the volume of a solid bounded above by a function of two variables over a rectangular region. By taking the limit, this becomes a double integral representing the volume of the solid.
- Properties of double integral are useful to simplify computation and find bounds on their values.
- We can use Fubini’s theorem to write and evaluate a double integral as an iterated integral.
- Double integrals are used to calculate the area of a region, the volume under a surface, and the average value of a function of two variables over a rectangular region.

Key Equations

- **Double integral**
  $$\iint_R f(x, y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$
- **Iterated integral**
  $$\int_a^b \int_c^d f(x, y)dx\, dy = \int_a^b \left[\int_c^d f(x, y)dy\right] dx$$
  or
  $$\int_c^d \int_b^a f(x, y)dx\, dy = \int_c^d \left[\int_a^b f(x, y)dx\right] dy$$
- **Average value of a function of two variables**
  $$f_{\text{ave}} = \frac{1}{\text{Area } R} \iint_R f(x, y)dx\, dy$$

In the following exercises, use the midpoint rule with $m = 4$ and $n = 2$ to estimate the volume of the solid bounded by the surface $z = f(x, y)$, the vertical planes $x = 1$, $x = 2$, $y = 1$, and $y = 2$, and the horizontal plane

$z = 0$.

Exercise:

Problem: $f(x, y) = 4x + 2y + 8xy$

Solution:

27.

Exercise:

Problem: $f(x, y) = 16x^2 + \frac{y}{2}$

In the following exercises, estimate the volume of the solid under the surface $z = f(x, y)$ and above the rectangular region *R* by using a Riemann sum with $m = n = 2$ and the sample points to be the lower left corners of the subrectangles of the partition.

Exercise:

Problem: $f(x, y) = \sin x - \cos y$, $R = [0, \pi] \times [0, \pi]$

Solution:

0.

Exercise:

Problem: $f(x, y) = \cos x + \cos y$, $R = [0, \pi] \times \left[0, \frac{\pi}{2}\right]$

Exercise:

Problem:

Use the midpoint rule with $m = n = 2$ to estimate $\iint\limits_R f(x, y)dA$, where the values of the function *f* on $R = [8, 10] \times [9, 11]$ are given in the following table.

	y				
x	9	9.5	10	10.5	11
8	9.8	5	6.7	5	5.6
8.5	9.4	4.5	8	5.4	3.4
9	8.7	4.6	6	5.5	3.4
9.5	6.7	6	4.5	5.4	6.7
10	6.8	6.4	5.5	5.7	6.8

Solution:

21.3.

Exercise:

Problem:

The values of the function *f* on the rectangle $R = [0, 2] \times [7, 9]$ are given in the following table. Estimate the double integral $\iint\limits_R f(x, y)dA$ by using a Riemann sum with $m = n = 2$. Select the sample points to be the upper right corners of the subsquares of *R*.

	$y_0 = 7$	$y_1 = 8$	$y_2 = 9$
$x_0 = 0$	10.22	10.21	9.85
$x_1 = 1$	6.73	9.75	9.63
$x_2 = 2$	5.62	7.83	8.21

Exercise:

Problem:

The depth of a children's 4-ft by 4-ft swimming pool, measured at 1-ft intervals, is given in the following table.

a. Estimate the volume of water in the swimming pool by using a Riemann sum with $m = n = 2$. Select the sample points using the midpoint rule on $R = [0, 4] \times [0, 4]$.
b. Find the average depth of the swimming pool.

	y				
x	0	1	2	3	4
0	1	1.5	2	2.5	3
1	1	1.5	2	2.5	3
2	1	1.5	1.5	2.5	3
3	1	1	1.5	2	2.5
4	1	1	1	1.5	2

Solution:

a. 28 ft^3 b. 1.75 ft.

Exercise:

Problem:

The depth of a 3-ft by 3-ft hole in the ground, measured at 1-ft intervals, is given in the following table.

a. Estimate the volume of the hole by using a Riemann sum with $m = n = 3$ and the sample points to be the upper left corners of the subsquares of *R*.
b. Find the average depth of the hole.

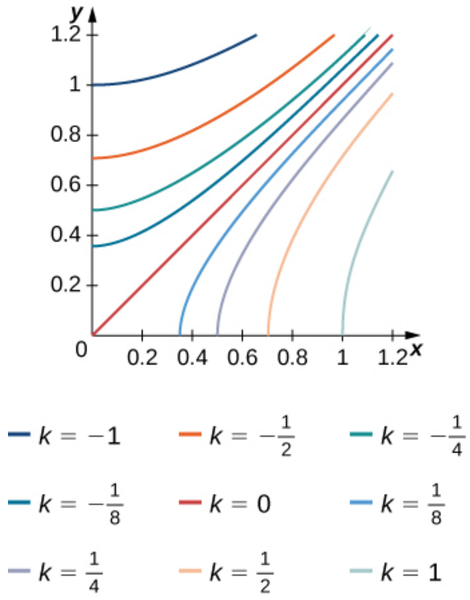
	y			
x	0	1	2	3
0	6	6.5	6.4	6
1	6.5	7	7.5	6.5
2	6.5	6.7	6.5	6
3	6	6.5	5	5.6

Exercise:

Problem:

The level curves $f(x, y) = k$ of the function *f* are given in the following graph, where *k* is a constant.

a. Apply the midpoint rule with $m = n = 2$ to estimate the double integral $\iint\limits_R f(x, y)dA$, where $R = [0.2, 1] \times [0, 0.8]$.
b. Estimate the average value of the function *f* on *R*.



Solution:

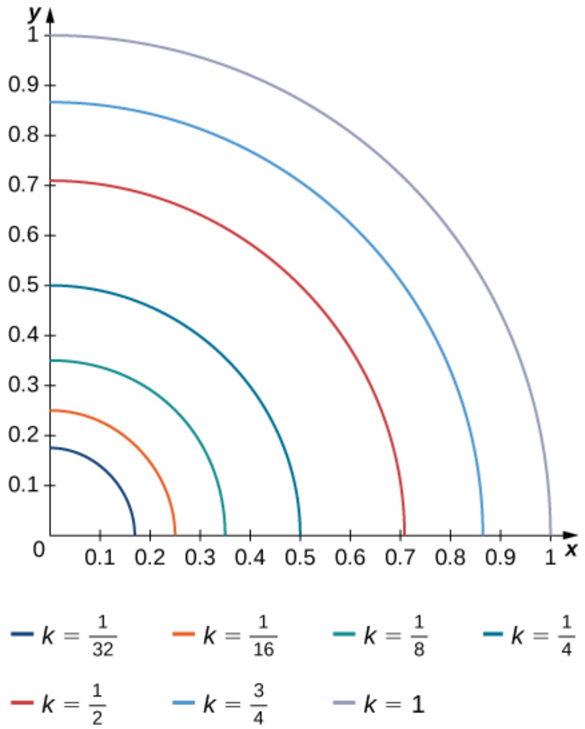
a. 0.112 b. $f_{\text{ave}} \simeq 0.175$; here $f(0.4, 0.2) \simeq 0.1$, $f(0.2, 0.6) \simeq -0.2$, $f(0.8, 0.2) \simeq 0.6$, and $f(0.8, 0.6) \simeq 0.2$.

Exercise:

Problem:

The level curves $f(x, y) = k$ of the function *f* are given in the following graph, where *k* is a constant.

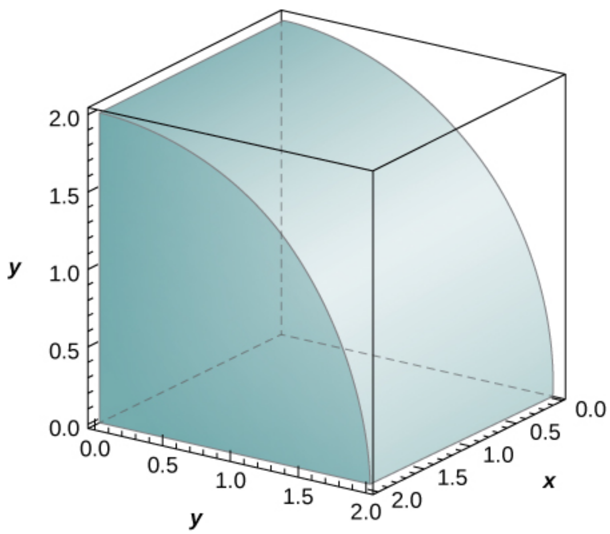
a. Apply the midpoint rule with $m = n = 2$ to estimate the double integral $\iint\limits_R f(x, y)dA$, where $R = [0.1, 0.5] \times [0.1, 0.5]$.
b. Estimate the average value of the function *f* on *R*.



Exercise:

Problem:

The solid lying under the surface $z = \sqrt{4 - y^2}$ and above the rectangular region $R = [0, 2] \times [0, 2]$ is illustrated in the following graph. Evaluate the double integral $\iint_R f(x, y) dA$, where $f(x, y) = \sqrt{4 - y^2}$, by finding the volume of the corresponding solid.



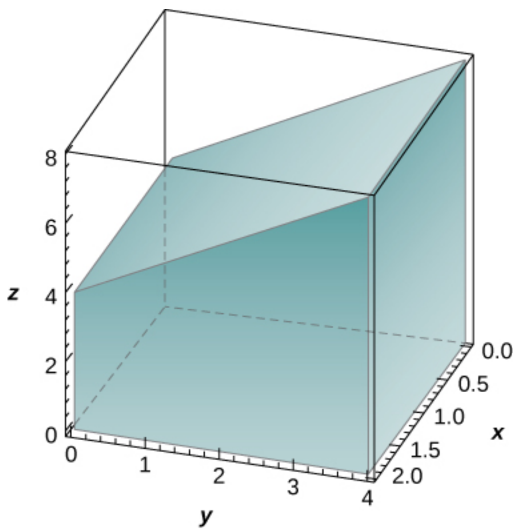
Solution:

2π.

Exercise:

Problem:

The solid lying under the plane $z = y + 4$ and above the rectangular region $R = [0, 2] \times [0, 4]$ is illustrated in the following graph. Evaluate the double integral $\iint\limits_R f(x, y)dA$, where $f(x, y) = y + 4$, by finding the volume of the corresponding solid.



In the following exercises, calculate the integrals by interchanging the order of integration.

Exercise:

Problem: $\displaystyle\int_{-1}^{1}\left(\int_{-2}^{2}(2x + 3y + 5)dx\right)dy$

Solution:

40.

Exercise:

Problem: $\displaystyle\int_{0}^{2}\left(\int_{0}^{1}(x + 2e^y - 3)dx\right)dy$

Exercise:

Problem: $\displaystyle\int_{1}^{27}\left(\int_{1}^{2}(\sqrt[3]{x} + \sqrt[3]{y})dy\right)dx$

Solution:

$\frac{81}{2} + 39\sqrt[3]{2}.$

Exercise:

Problem: $\int\limits_1^{16}\left(\int\limits_1^{8}\left(\sqrt[4]{x}+2\sqrt[3]{y}\right)dy\right)dx$

Exercise:

Problem: $\int\limits_{\ln 2}^{\ln 3}\left(\int\limits_0^{1}e^{x+y}dy\right)dx$

Solution:

$e-1.$

Exercise:

Problem: $\int\limits_0^{2}\left(\int\limits_0^{1}3^{x+y}dy\right)dx$

Exercise:

Problem: $\int\limits_1^{6}\left(\int\limits_2^{9}\frac{\sqrt{y}}{x^2}dy\right)dx$

Solution:

$15-\frac{10\sqrt{2}}{9}.$

Exercise:

Problem: $\int\limits_1^{9}\left(\int\limits_4^{2}\frac{\sqrt{x}}{y^2}dy\right)dx$

In the following exercises, evaluate the iterated integrals by choosing the order of integration.

Exercise:

Problem: $\int\limits_0^{\pi}\int\limits_0^{\pi/2}\sin(2x)\cos(3y)dx\,dy$

Solution:

$0.$

Exercise:

Problem: $\displaystyle\int_{\pi/12}^{\pi/8}\int_{\pi/4}^{\pi/3}[\cot x+\tan(2y)]dx\,dy$

Exercise:

Problem: $\displaystyle\int_{1}^{e}\int_{1}^{e}\left[\frac{1}{x}\sin(\ln x)+\frac{1}{y}\cos(\ln y)\right]dx\,dy$

Solution:

$(e-1)(1+\sin 1-\cos 1).$

Exercise:

Problem: $\displaystyle\int_{1}^{e}\int_{1}^{e}\frac{\sin(\ln x)\cos(\ln y)}{xy}dx\,dy$

Exercise:

Problem: $\displaystyle\int_{1}^{2}\int_{1}^{2}\left(\frac{\ln y}{x}+\frac{x}{2y+1}\right)dy\,dx$

Solution:

$\frac{3}{4}\ln\left(\frac{5}{3}\right)+2\ln^2 2-\ln 2.$

Exercise:

Problem: $\displaystyle\int_{1}^{e}\int_{1}^{2}x^2\ln(x)dy\,dx$

Exercise:

Problem: $\displaystyle\int_{1}^{\sqrt{3}}\int_{1}^{2}y\arctan\left(\frac{1}{x}\right)dy\,dx$

Solution:

$\frac{1}{8}\left[\left(2\sqrt{3}-3\right)\pi+6\ln 2\right].$

Exercise:

Problem: $\displaystyle\int_{0}^{1}\int_{0}^{1/2}(\arcsin x+\arcsin y)dy\,dx$

Exercise:

Problem: $\int\limits_0^1 \int\limits_1^2 xe^{x+4y} dy\, dx$

Solution:

$\frac{1}{4}e^4\left(e^4 - 1\right).$

Exercise:

Problem: $\int\limits_1^2 \int\limits_0^1 xe^{x-y} dy\, dx$

Exercise:

Problem: $\int\limits_1^e \int\limits_1^e \left(\frac{\ln y}{\sqrt{y}} + \frac{\ln x}{\sqrt{x}}\right) dy\, dx$

Solution:

$4(e-1)\left(2 - \sqrt{e}\right).$

Exercise:

Problem: $\int\limits_1^e \int\limits_1^e \left(\frac{x \ln y}{\sqrt{y}} + \frac{y \ln x}{\sqrt{x}}\right) dy\, dx$

Exercise:

Problem: $\int\limits_0^1 \int\limits_1^2 \left(\frac{x}{x^2 + y^2}\right) dy\, dx$

Solution:

$-\frac{\pi}{4} + \ln\left(\frac{5}{4}\right) - \frac{1}{2}\ln 2 + \arctan 2.$

Exercise:

Problem: $\int\limits_0^1 \int\limits_1^2 \frac{y}{x + y^2} dy\, dx$

In the following exercises, find the average value of the function over the given rectangles.

Exercise:

Problem: $f(x,y) = -x + 2y$, $R = [0,1] \times [0,1]$

Solution:

$\frac{1}{2}$.

Exercise:

Problem: $f(x,y)=x^4+2y^3$, $R=[1,2]\times[2,3]$

Exercise:

Problem: $f(x,y)=\sinh x+\sinh y$, $R=[0,1]\times[0,2]$

Solution:

$\frac{1}{2}(2\cosh 1+\cosh 2-3)$.

Exercise:

Problem: $f(x,y)=\arctan(xy)$, $R=[0,1]\times[0,1]$

Exercise:

Problem:

Let f and g be two continuous functions such that $0\le m_1\le f(x)\le M_1$ for any $x\in[a,b]$ and $0\le m_2\le g(y)\le M_2$ for any $y\in[c,d]$. Show that the following inequality is true:

$$m_1m_2(b-a)(c-d)\le\int_a^b\int_c^d f(x)g(y)dy\,dx\le M_1M_2(b-a)(c-d).$$

In the following exercises, use property v. of double integrals and the answer from the preceding exercise to show that the following inequalities are true.

Exercise:

Problem: $\frac{1}{e^2}\le\iint_R e^{-x^2-y^2}dA\le 1$, where $R=[0,1]\times[0,1]$

Exercise:

Problem: $\frac{\pi^2}{144}\le\iint_R\sin x\cos y\,dA\le\frac{\pi^2}{48}$, where $R=\left[\frac{\pi}{6},\frac{\pi}{3}\right]\times\left[\frac{\pi}{6},\frac{\pi}{3}\right]$

Exercise:

Problem: $0\le\iint_R e^{-y}\cos x\,dA\le\frac{\pi}{2}$, where $R=\left[0,\frac{\pi}{2}\right]\times\left[0,\frac{\pi}{2}\right]$

Exercise:

Problem: $0\le\iint_R(\ln x)(\ln y)dA\le(e-1)^2$, where $R=[1,e]\times[1,e]$

Exercise:

Problem:

Let *f* and *g* be two continuous functions such that $0 \le m_1 \le f(x) \le M_1$ for any $x \in [a, b]$ and $0 \le m_2 \le g(y) \le M_2$ for any $y \in [c, d]$. Show that the following inequality is true:

$$(m_1 + m_2)(b-a)(c-d) \le \int_a^b \int_c^d [f(x) + g(y)]dy\,dx \le (M_1 + M_2)(b-a)(c-d).$$

In the following exercises, use property v. of double integrals and the answer from the preceding exercise to show that the following inequalities are true.

Exercise:

Problem: $\frac{2}{e} \le \iint\limits_R \left(e^{-x^2} + e^{-y^2}\right)dA \le 2$, where $R = [0, 1] \times [0, 1]$

Exercise:

Problem: $\frac{\pi^2}{36} \le \iint\limits_R (\sin x + \cos y)dA \le \frac{\pi^2\sqrt{3}}{36}$, where $R = \left[\frac{\pi}{6}, \frac{\pi}{3}\right] \times \left[\frac{\pi}{6}, \frac{\pi}{3}\right]$

Exercise:

Problem: $\frac{\pi}{2}e^{-\pi/2} \le \iint\limits_R \left(\cos x + e^{-y}\right)dA \le \pi$, where $R = \left[0, \frac{\pi}{2}\right] \times \left[0, \frac{\pi}{2}\right]$

Exercise:

Problem: $\frac{1}{e} \le \iint\limits_R \left(e^{-y} - \ln x\right)dA \le 2$, where $R = [0, 1] \times [0, 1]$

In the following exercises, the function *f* is given in terms of double integrals.

a. Determine the explicit form of the function *f*.
b. Find the volume of the solid under the surface $z = f(x, y)$ and above the region *R*.
c. Find the average value of the function *f* on *R*.
d. Use a computer algebra system (CAS) to plot $z = f(x, y)$ and $z = f_{\text{ave}}$ in the same system of coordinates.

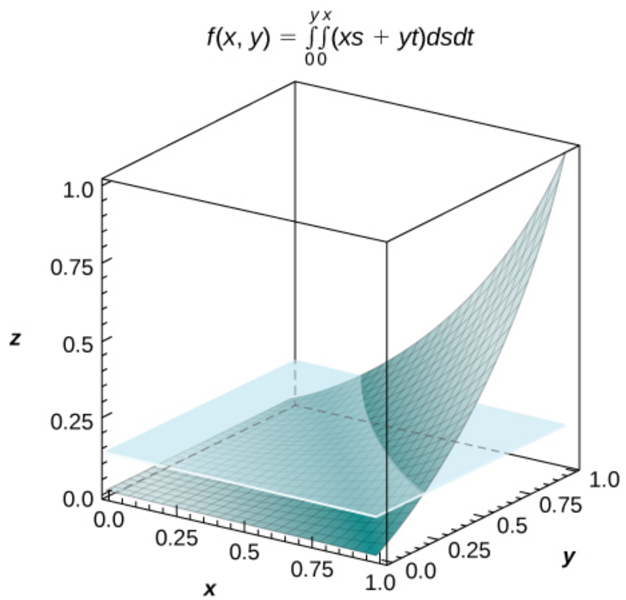
Exercise:

Problem: **[T]** $f(x, y) = \int_0^y \int_0^x (xs + yt)ds\,dt$, where $(x, y) \in R = [0, 1] \times [0, 1]$

Solution:

a. $f(x, y) = \frac{1}{2}xy\left(x^2 + y^2\right)$ b. $V = \int_0^1 \int_0^1 f(x, y)dx\,dy = \frac{1}{8}$ c. $f_{\text{ave}} = \frac{1}{8}$;

d.



Exercise:

Problem: **[T]** $f(x,y) = \int_0^x \int_0^y [\cos(s) + \cos(t)]dt\, ds$, where $(x,y) \in R = [0,3] \times [0,3]$

Exercise:

Problem: Show that if *f* and *g* are continuous on $[a,b]$ and $[c,d]$, respectively, then

$$\int_a^b \int_c^d [f(x) + g(y)]dy\, dx = (d-c)\int_a^b f(x)dx$$

$$+\int_a^b \int_c^d g(y)dy\, dx = (b-a)\int_c^d g(y)dy + \int_c^d \int_a^b f(x)dx\, dy.$$

Exercise:

Problem:

Show that $\int_a^b \int_c^d yf(x) + xg(y)dy\, dx = \frac{1}{2}(d^2-c^2)\left(\int_a^b f(x)dx\right) + \frac{1}{2}(b^2-a^2)\left(\int_c^d g(y)dy\right).$

Exercise:

Problem: **[T]** Consider the function $f(x,y) = e^{-x^2-y^2}$, where $(x,y) \in R = [-1,1] \times [-1,1]$.

a. Use the midpoint rule with $m = n = 2, 4, \ldots, 10$ to estimate the double integral $I = \iint_R e^{-x^2-y^2} dA$.

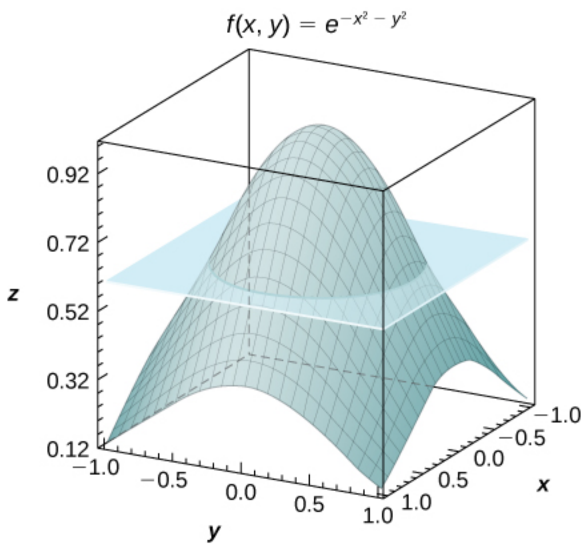
Round your answers to the nearest hundredths.

b. For $m = n = 2$, find the average value of *f* over the region *R*. Round your answer to the nearest hundredths.

c. Use a CAS to graph in the same coordinate system the solid whose volume is given by $\iint_R e^{-x^2-y^2} dA$ and the plane $z = f_{\text{ave}}$.

Solution:

a. For $m = n = 2$, $I = 4e^{-0.5} \approx 2.43$ b. $f_{\text{ave}} = e^{-0.5} \simeq 0.61$;
c.



Exercise:

Problem: **[T]** Consider the function $f(x, y) = \sin(x^2)\cos(y^2)$, where $(x, y) \in R = [-1, 1] \times [-1, 1]$.

a. Use the midpoint rule with $m = n = 2, 4, \ldots, 10$ to estimate the double integral $I = \iint_R \sin(x^2)\cos(y^2) dA$. Round your answers to the nearest hundredths.

b. For $m = n = 2$, find the average value of *f* over the region *R*. Round your answer to the nearest hundredths.

c. Use a CAS to graph in the same coordinate system the solid whose volume is given by $\iint_R \sin(x^2)\cos(y^2) dA$ and the plane $z = f_{\text{ave}}$.

In the following exercises, the functions f_n are given, where $n \geq 1$ is a natural number.

a. Find the volume of the solids S_n under the surfaces $z = f_n(x, y)$ and above the region *R*.
b. Determine the limit of the volumes of the solids S_n as *n* increases without bound.

Exercise:

Problem: $f(x, y) = x^n + y^n + xy, (x, y) \in R = [0, 1] \times [0, 1]$

Solution:

a. $\frac{2}{n+1} + \frac{1}{4}$ b. $\frac{1}{4}$

Exercise:

Problem: $f(x, y) = \frac{1}{x^n} + \frac{1}{y^n}, (x, y) \in R = [1, 2] \times [1, 2]$

Exercise:

Problem:

Show that the average value of a function *f* on a rectangular region $R = [a, b] \times [c, d]$ is $f_{\text{ave}} \approx \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(x_{ij}^{*}, y_{ij}^{*}\right)$, where $\left(x_{ij}^{*}, y_{ij}^{*}\right)$ are the sample points of the partition of *R*, where $1 \leq i \leq m$ and $1 \leq j \leq n$.

Exercise:

Problem:

Use the midpoint rule with $m = n$ to show that the average value of a function *f* on a rectangular region $R = [a, b] \times [c, d]$ is approximated by

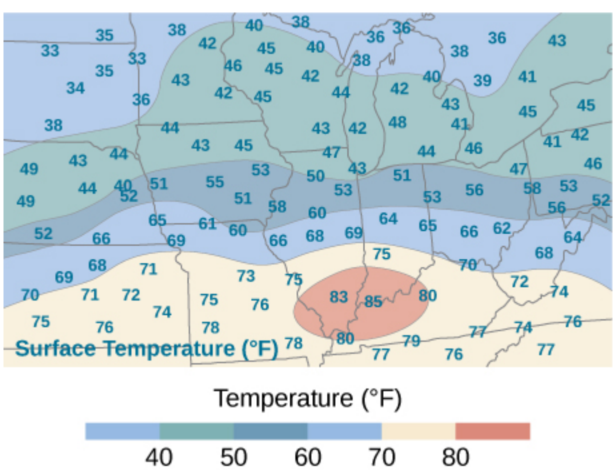
Equation:

$$f_{\text{ave}} \approx \frac{1}{n^2} \sum_{i,j=1}^{n} f\left(\frac{1}{2}(x_{i-1} + x_i), \frac{1}{2}(y_{j-1} + y_j)\right).$$

Exercise:

Problem:

An isotherm map is a chart connecting points having the same temperature at a given time for a given period of time. Use the preceding exercise and apply the midpoint rule with $m = n = 2$ to find the average temperature over the region given in the following figure.



Solution:

56.5° F; here $f(x_1^*, y_1^*) = 71$, $f(x_2^*, y_1^*) = 72$, $f(x_2^*, y_1^*) = 40$, $f(x_2^*, y_2^*) = 43$, where x_i^* and y_j^* are the midpoints of the subintervals of the partitions of $[a, b]$ and $[c, d]$, respectively.

Glossary

double integral

of the function $f(x, y)$ over the region R in the xy-plane is defined as the limit of a double Riemann sum, $\iint\limits_R f(x, y)dA = \lim\limits_{m,n\to\infty} \sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A.$

double Riemann sum

of the function $f(x, y)$ over a rectangular region R is $\sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$ where R is divided into smaller subrectangles R_{ij} and (x_{ij}^*, y_{ij}^*) is an arbitrary point in R_{ij}

Fubini's theorem

if $f(x, y)$ is a function of two variables that is continuous over a rectangular region $R = \{(x, y) \in \mathbb{R}^2 | a \le x \le b, c \le y \le d\}$, then the double integral of f over the region equals an iterated integral, $\iint\limits_R f(x, y)dy\,dx = \int_a^b \int_c^d f(x, y)dx\,dy = \int_c^d \int_a^b f(x, y)dx\,dy$

iterated integral

for a function $f(x, y)$ over the region R is

a. $\int_a^b \int_c^d f(x, y)dx\,dy = \int_a^b \left[\int_c^d f(x, y)dy\right]dx,$

b. $\int_c^d \int_b^a f(x, y)dx\,dy = \int_c^d \left[\int_a^b f(x, y)dx\right]dy,$

where $a, b, c,$ and d are any real numbers and $R = [a, b] \times [c, d]$

Double Integrals over General Regions

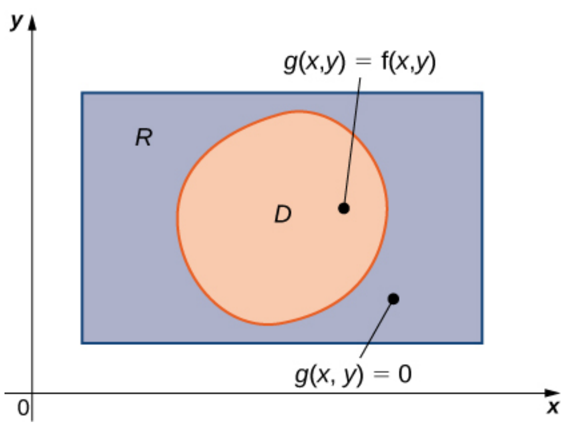
- Recognize when a function of two variables is integrable over a general region.
- Evaluate a double integral by computing an iterated integral over a region bounded by two vertical lines and two functions of x, or two horizontal lines and two functions of y.
- Simplify the calculation of an iterated integral by changing the order of integration.
- Use double integrals to calculate the volume of a region between two surfaces or the area of a plane region.
- Solve problems involving double improper integrals.

In Double Integrals over Rectangular Regions, we studied the concept of double integrals and examined the tools needed to compute them. We learned techniques and properties to integrate functions of two variables over rectangular regions. We also discussed several applications, such as finding the volume bounded above by a function over a rectangular region, finding area by integration, and calculating the average value of a function of two variables.

In this section we consider double integrals of functions defined over a general bounded region D on the plane. Most of the previous results hold in this situation as well, but some techniques need to be extended to cover this more general case.

General Regions of Integration

An example of a general bounded region D on a plane is shown in [link]. Since D is bounded on the plane, there must exist a rectangular region R on the same plane that encloses the region D, that is, a rectangular region R exists such that D is a subset of $R\,(D \subseteq R)$.



For a region D that is a subset of R, we can define a function $g(x,y)$ to equal $f(x,y)$ at every point in D and 0 at every point of R not in D.

Suppose $z = f\,(x,y)$ is defined on a general planar bounded region D as in [link]. In order to develop double integrals of f over D, we extend the definition of the function to include all points on the rectangular region R and then use the concepts and tools from the preceding section. But how do we extend the definition of f to include all the points on R? We do this by defining a new function $g\,(x,y)$ on R as follows:

Equation:

$$g(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \text{ is in } D \\ 0 & \text{if } (x,y) \text{ is in } R \text{ but not in } D \end{cases}$$

Note that we might have some technical difficulties if the boundary of D is complicated. So we assume the boundary to be a piecewise smooth and continuous simple closed curve. Also, since all the results developed in Double Integrals over Rectangular Regions used an integrable function $f(x,y)$, we must be careful about $g(x,y)$ and verify that $g(x,y)$ is an integrable function over the rectangular region R. This happens as long as the region D is bounded by simple closed curves. For now we will concentrate on the descriptions of the regions rather than the function and extend our theory appropriately for integration.

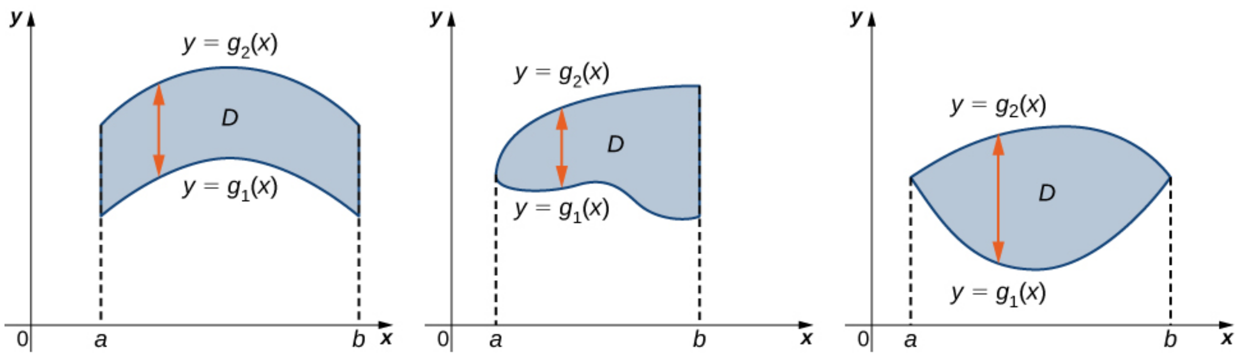
We consider two types of planar bounded regions.

Note:
Definition
A region D in the (x,y)-plane is of **Type I** if it lies between two vertical lines and the graphs of two continuous functions $g_1(x)$ and $g_2(x)$. That is ([link]),
Equation:

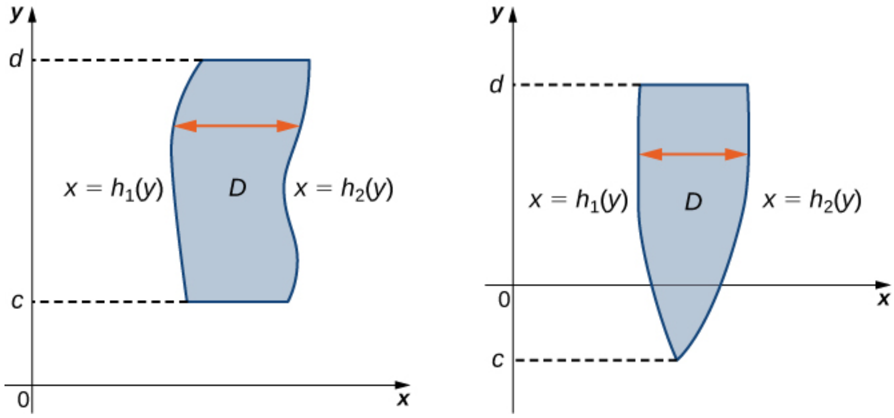
$$D = \{(x,y) | a \le x \le b, g_1(x) \le y \le g_2(x)\}.$$

A region D in the xy plane is of **Type II** if it lies between two horizontal lines and the graphs of two continuous functions $h_1(y)$ and $h_2(y)$. That is ([link]),
Equation:

$$D = \{(x,y) | c \le y \le d, h_1(y) \le x \le h_2(y)\}.$$



A Type I region lies between two vertical lines and the graphs of two functions of x.

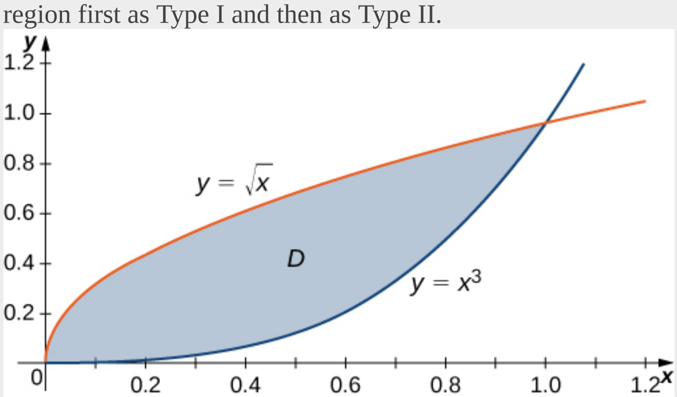


A Type II region lies between two horizontal lines and the graphs of two functions of y.

Example:
Exercise:

Problem:
Describing a Region as Type I and Also as Type II

Consider the region in the first quadrant between the functions $y = \sqrt{x}$ and $y = x^3$ ([link]). Describe the region first as Type I and then as Type II.



Region D can be described as Type I or as Type II.

Solution:

When describing a region as Type I, we need to identify the function that lies above the region and the function that lies below the region. Here, region D is bounded above by $y = \sqrt{x}$ and below by $y = x^3$ in the interval for x in $[0, 1]$. Hence, as Type I, D is described as the set $\{(x, y)|0 \leq x \leq 1, x^3 \leq y \leq \sqrt{x}\}$.

However, when describing a region as Type II, we need to identify the function that lies on the left of the region and the function that lies on the right of the region. Here, the region D is bounded on the left by

$x = y^2$ and on the right by $x = \sqrt[3]{y}$ in the interval for y in $[0, 1]$. Hence, as Type II, D is described as the set $\{(x, y)|0 \leq y \leq 1, y^2 \leq x \leq \sqrt[3]{y}\}$.

Note:
Exercise:

Problem:

Consider the region in the first quadrant between the functions $y = 2x$ and $y = x^2$. Describe the region first as Type I and then as Type II.

Solution:

Type I and Type II are expressed as $\{(x, y)|0 \leq x \leq 2, x^2 \leq y \leq 2x\}$ and $\{(x, y)|0 \leq y \leq 4, \frac{1}{2}y \leq x \leq \sqrt{y}\}$, respectively.

Hint

Graph the functions, and draw vertical and horizontal lines.

Double Integrals over Nonrectangular Regions

To develop the concept and tools for evaluation of a double integral over a general, nonrectangular region, we need to first understand the region and be able to express it as Type I or Type II or a combination of both. Without understanding the regions, we will not be able to decide the limits of integrations in double integrals. As a first step, let us look at the following theorem.

Note:
Double Integrals over Nonrectangular Regions
Suppose $g(x, y)$ is the extension to the rectangle R of the function $f(x, y)$ defined on the regions D and R as shown in [link] inside R. Then $g(x, y)$ is integrable and we define the double integral of $f(x, y)$ over D by
Equation:

$$\iint_D f(x, y) dA = \iint_R g(x, y) dA.$$

The right-hand side of this equation is what we have seen before, so this theorem is reasonable because R is a rectangle and $\iint_R g(x, y) dA$ has been discussed in the preceding section. Also, the equality works because the values of $g(x, y)$ are 0 for any point (x, y) that lies outside D, and hence these points do not add anything to the integral. However, it is important that the rectangle R contains the region D.

As a matter of fact, if the region D is bounded by smooth curves on a plane and we are able to describe it as Type I or Type II or a mix of both, then we can use the following theorem and not have to find a rectangle R containing the region.

Note:
Fubini’s Theorem (Strong Form)
For a function $f\left(x,y\right)$ that is continuous on a region D of Type I, we have
Equation:

$$\iint\limits_D f(x,y)dA = \iint\limits_D f(x,y)dy\,dx = \int\limits_a^b \left[\int\limits_{g_1(x)}^{g_2(x)} f(x,y)dy\right]dx.$$

Similarly, for a function $f\left(x,y\right)$ that is continuous on a region D of Type II, we have
Equation:

$$\iint\limits_D f(x,y)dA = \iint\limits_D f(x,y)dx\,dy = \int\limits_c^d \left[\int\limits_{h_1(y)}^{h_2(y)} f(x,y)dx\right]dy.$$

The integral in each of these expressions is an iterated integral, similar to those we have seen before. Notice that, in the inner integral in the first expression, we integrate $f\left(x,y\right)$ with x being held constant and the limits of integration being $g_1\left(x\right)$ and $g_2\left(x\right)$. In the inner integral in the second expression, we integrate $f\left(x,y\right)$ with y being held constant and the limits of integration are $h_1\left(x\right)$ and $h_2\left(x\right)$.

Example:
Exercise:

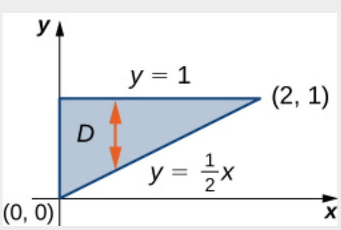
Problem:
Evaluating an Iterated Integral over a Type I Region

Evaluate the integral $\iint\limits_D x^2e^{xy}dA$ where D is shown in [link].

Solution:

First construct the region D as a Type I region ([link]). Here $D = \left\{(x,y)|0 \le x \le 2, \frac{1}{2}x \le y \le 1\right\}$. Then we have
Equation:

$$\iint\limits_D x^2e^{xy}dA = \int\limits_{x=0}^{x=2} \int\limits_{y=1/2x}^{y=1} x^2e^{xy}dy\,dx.$$



We can express region D

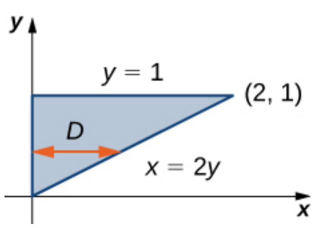
as a Type I region and integrate from $y=\frac{1}{2}x$ to $y=1$, between the lines $x=0$ and $x=2$.

Therefore, we have

Equation:

$$\begin{aligned}
\int_{x=0}^{x=2}\int_{y=\frac{1}{2}x}^{y=1} x^2 e^{xy}\,dy\,dx &= \int_{x=0}^{x=2}\left[\int_{y=1/2x}^{y=1} x^2 e^{xy}\,dy\right]dx && \text{Iterated integral for a Type I region.} \\
&= \int_{x=0}^{x=2}\left[x^2\frac{e^{xy}}{x}\right]\Big|_{y=1/2x}^{y=1} dx && \text{Integrate with respect to } y \text{ using } u\text{-substitution with } u=xy \text{ where } x \text{ is held constant.} \\
&= \int_{x=0}^{x=2}\left[xe^x - xe^{x^2/2}\right]dx && \text{Integrate with respect to } x \text{ using } u\text{-substitution with } u=\tfrac{1}{2}x^2. \\
&= \left[xe^x - e^x - e^{\frac{1}{2}x^2}\right]\Big|_{x=0}^{x=2} = 2
\end{aligned}$$

In [link], we could have looked at the region in another way, such as $D=\{(x,y)|0\le y\le 1, 0\le x\le 2y\}$ ([link]).



This is a Type II region and the integral would then look like

Equation:

$$\iint_D x^2 e^{xy}\,dA = \int_{y=0}^{y=1}\int_{x=0}^{x=2y} x^2 e^{xy}\,dx\,dy.$$

However, if we integrate first with respect to x, this integral is lengthy to compute because we have to use integration by parts twice.

Example:

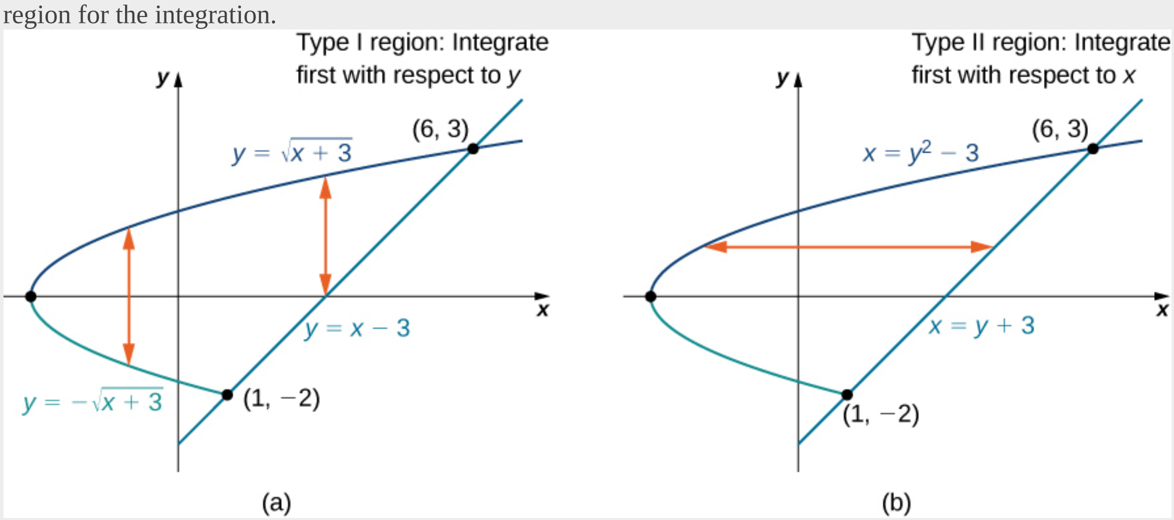
Exercise:

Problem:
Evaluating an Iterated Integral over a Type II Region

Evaluate the integral $\iint_D (3x^2 + y^2)\,dA$ where $= \{(x, y)| -2 \le y \le 3, y^2 - 3 \le x \le y + 3\}$.

Solution:

Notice that D can be seen as either a Type I or a Type II region, as shown in [link]. However, in this case describing D as Type I is more complicated than describing it as Type II. Therefore, we use D as a Type II region for the integration.



The region D in this example can be either (a) Type I or (b) Type II.

Choosing this order of integration, we have

Equation:

$$\begin{aligned}
\iint_D (3x^2 + y^2)\,dA &= \int_{y=-2}^{y=3} \int_{x=y^2-3}^{x=y+3} (3x^2 + y^2)\,dx\,dy && \text{Iterated integral, T} \\
&= \int_{y=-2}^{y=3} \left(x^3 + xy^2\right)\Big|_{y^2-3}^{y+3} dy && \text{Integrate with resp} \\
&= \int_{y=-2}^{y=3} \left((y+3)^3 + (y+3)y^2 - (y^2-3)^3 - (y^2-3)y^2\right) dy \\
&= \int_{-2}^{3} \left(54 + 27y - 12y^2 + 2y^3 + 8y^4 - y^6\right) dy && \text{Integrate with resp} \\
&= \left[54y + \tfrac{27y^2}{2} - 4y^3 + \tfrac{y^4}{2} + \tfrac{8y^5}{5} - \tfrac{y^7}{7}\right]\Big|_{-2}^{3} \\
&= \tfrac{2375}{7}.
\end{aligned}$$

Note:
Exercise:

Problem:

Sketch the region D and evaluate the iterated integral $\iint\limits_D xy\,dy\,dx$ where D is the region bounded by the curves $y = \cos x$ and $y = \sin x$ in the interval $[-3\pi/4, \pi/4]$.

Solution:

$\pi/4$

Hint

Express D as a Type I region, and integrate with respect to y first.

Recall from Double Integrals over Rectangular Regions the properties of double integrals. As we have seen from the examples here, all these properties are also valid for a function defined on a nonrectangular bounded region on a plane. In particular, property 3 states:

If $R = S \cup T$ and $S \cap T = \emptyset$ except at their boundaries, then
Equation:

$$\iint\limits_R f(x,y)dA = \iint\limits_S f(x,y)dA + \iint\limits_T f(x,y)dA.$$

Similarly, we have the following property of double integrals over a nonrectangular bounded region on a plane.

Note:
Decomposing Regions into Smaller Regions
Suppose the region D can be expressed as $D = D_1 \cup D_2$ where D_1 and D_2 do not overlap except at their boundaries. Then
Equation:

$$\iint\limits_D f(x,y)dA = \iint\limits_{D_1} f(x,y)dA + \iint\limits_{D_2} f(x,y)dA.$$

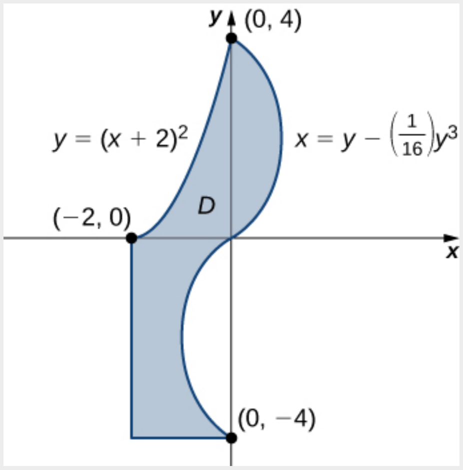
This theorem is particularly useful for nonrectangular regions because it allows us to split a region into a union of regions of Type I and Type II. Then we can compute the double integral on each piece in a convenient way, as in the next example.

Example:
Exercise:

Problem:
Decomposing Regions

Express the region D shown in [link] as a union of regions of Type I or Type II, and evaluate the integral
Equation:

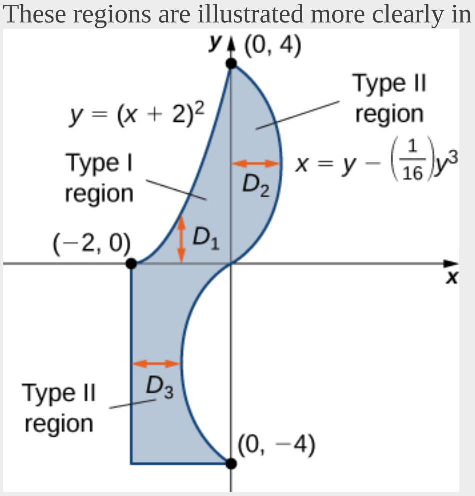
$$\iint_D (2x + 5y)dA.$$



This region can be decomposed into a union of three regions of Type I or Type II.

Solution:

The region D is not easy to decompose into any one type; it is actually a combination of different types. So we can write it as a union of three regions D_1, D_2, and D_3 where,
$D_1 = \left\{(x,y) | -2 \le x \le 0, 0 \le y \le (x+2)^2\right\}, D_2 = \left\{(x,y) | 0 \le y \le 4, 0 \le x \le \left(y - \frac{1}{16}y^3\right)\right\}.$
These regions are illustrated more clearly in [link].



Breaking the region into three

subregions makes it easier to set up the integration.

Here D_1 is Type I and D_2 and D_3 are both of Type II. Hence,

Equation:

$$\begin{aligned}
\iint\limits_{D} (2x+5y)dA &= \iint\limits_{D_1} (2x+5y)dA + \iint\limits_{D_2} (2x+5y)dA + \iint\limits_{D_3} (2x+5y)dA \\
&= \int\limits_{x=-2}^{x=0} \int\limits_{y=0}^{y=(x+2)^2} (2x+5y)dy\,dx + \int\limits_{y=0}^{y=4} \int\limits_{x=0}^{x=y-(1/16)y^3} (2+5y)dx\,dy + \int\limits_{y=-4}^{y=0} \int\limits_{x=-2}^{x=y-(1/16)y^3} (\\
&= \int\limits_{x=-2}^{x=0} \left[\frac{1}{2}(2+x)^2(20+24x+5x^2)\right] + \int\limits_{y=0}^{y=4} \left[\frac{1}{256}y^6 - \frac{7}{16}y^4 + 6y^2\right] \\
&\quad + \int\limits_{y=-4}^{y=0} \left[\frac{1}{256}y^6 - \frac{7}{16}y^4 + 6y^2 + 10y - 4\right] \\
&= \tfrac{40}{3} + \tfrac{1664}{35} - \tfrac{1696}{35} = \tfrac{1304}{105}.
\end{aligned}$$

Now we could redo this example using a union of two Type II regions (see the Checkpoint).

Note:
Exercise:

Problem:

Consider the region bounded by the curves $y = \ln x$ and $y = e^x$ in the interval $[1, 2]$. Decompose the region into smaller regions of Type II.

Solution:

$\{(x,y)|0 \le y \le 1, 1 \le x \le e^y\} \cup \{(x,y)|1 \le y \le e, 1 \le x \le 2\} \cup \{(x,y)|e \le y \le e^2, \ln y \le x \le 2\}$

Hint

Sketch the region, and split it into three regions to set it up.

Note:
Exercise:

Problem: Redo [link] using a union of two Type II regions.

Solution:

Same as in the example shown.

Hint

$\left\{(x, y) | 0 \leq y \leq 4, 2+\sqrt{y} \leq x \leq \left(y-\frac{1}{16} y^{3}\right)\right\} \cup \left\{(x, y) | -4 \leq y \leq 0, -2 \leq x \leq \left(y-\frac{1}{16} y^{3}\right)\right\}$

Changing the Order of Integration

As we have already seen when we evaluate an iterated integral, sometimes one order of integration leads to a computation that is significantly simpler than the other order of integration. Sometimes the order of integration does not matter, but it is important to learn to recognize when a change in order will simplify our work.

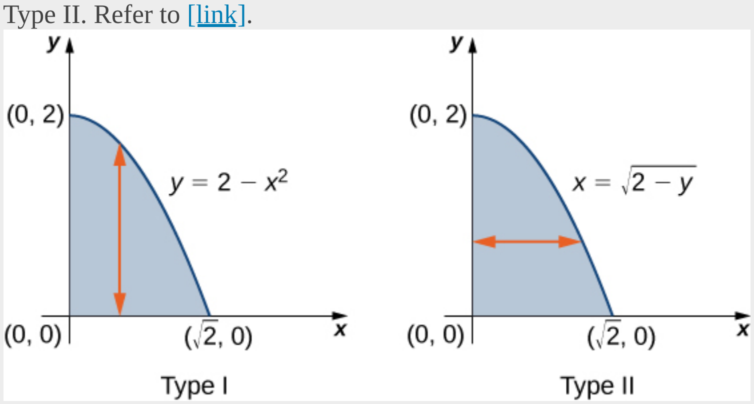
Example:
Exercise:

Problem:
Changing the Order of Integration

Reverse the order of integration in the iterated integral $\int_{x=0}^{x=\sqrt{2}} \int_{y=0}^{y=2-x^2} x e^{x^2} \, dy \, dx$. Then evaluate the new iterated integral.

Solution:

The region as presented is of Type I. To reverse the order of integration, we must first express the region as Type II. Refer to [link].



Converting a region from Type I to Type II.

We can see from the limits of integration that the region is bounded above by $y = 2 - x^2$ and below by $y = 0$, where x is in the interval $\left[0, \sqrt{2}\right]$. By reversing the order, we have the region bounded on the left by $x = 0$ and on the right by $x = \sqrt{2 - y}$ where y is in the interval $[0, 2]$. We solved $y = 2 - x^2$ in terms of x to obtain $x = \sqrt{2 - y}$.

Hence
Equation:

$$\int_0^{\sqrt{2}} \int_0^{2-x^2} xe^{x^2} dy\, dx = \int_0^2 \int_0^{\sqrt{2-y}} xe^{x^2} dx\, dy$$
$$= \int_0^2 \left[\frac{1}{2} ex^2 \Big|_0^{\sqrt{2-y}} \right] dy = \int_0^2 \frac{1}{2}\left(e^{2-y} - 1\right) dy = -\tfrac{1}{2}\left(e^{2-y} + y\right)\Big|_0^2$$
$$= \tfrac{1}{2}\left(e^2 - 3\right).$$

Reverse the ord integration the substitution.

Example:
Exercise:

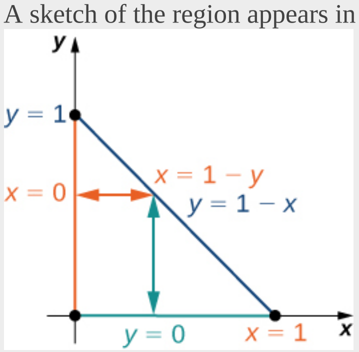
Problem:
Evaluating an Iterated Integral by Reversing the Order of Integration

Consider the iterated integral $\iint\limits_R f(x,y) dx\, dy$ where $z = f(x,y) = x - 2y$ over a triangular region R that has sides on $x = 0, y = 0$, and the line $x + y = 1$. Sketch the region, and then evaluate the iterated integral by

a. integrating first with respect to y and then
b. integrating first with respect to x.

Solution:

A sketch of the region appears in [link].



A triangular region R for integrating in two ways.

We can complete this integration in two different ways.

a. One way to look at it is by first integrating y from $y = 0$ to $y = 1 - x$ vertically and then integrating x from $x = 0$ to $x = 1$:
Equation:

$$\iint_R f(x,y)dx\,dy = \int_{x=0}^{x=1}\int_{y=0}^{y=1-x}(x-2y)dy\,dx = \int_{x=0}^{x=1}\left[xy-2y^2\right]_{y=0}^{y=1-x}dx$$

$$= \int_{x=0}^{x=1}\left[x(1-x)-(1-x)^2\right]dx = \int_{x=0}^{x=1}\left[-1+3x-2x^2\right]dx = \left[-x+\frac{3}{2}x^2-\frac{2}{3}\right.$$

b. The other way to do this problem is by first integrating x from $x=0$ to $x=1-y$ horizontally and then integrating y from $y=0$ to $y=1$:
Equation:

$$\iint_R f(x,y)dx\,dy = \int_{y=0}^{y=1}\int_{x=0}^{x=1-y}(x-2y)dx\,dy = \int_{y=0}^{y=1}\left[\frac{1}{2}x^2-2xy\right]_{x=0}^{x=1-y}dy$$

$$= \int_{y=0}^{y=1}\left[\frac{1}{2}(1-y)^2-2y(1-y)\right]dy = \int_{y=0}^{y=1}\left[\frac{1}{2}-3y+\frac{5}{2}y^2\right]dy$$

$$= \left[\tfrac{1}{2}y-\tfrac{3}{2}y^2+\tfrac{5}{6}y^3\right]_{y=0}^{y=1} = -\tfrac{1}{6}.$$

Note:
Exercise:

Problem:

Evaluate the iterated integral $\iint_D (x^2+y^2)dA$ over the region D in the first quadrant between the functions $y=2x$ and $y=x^2$. Evaluate the iterated integral by integrating first with respect to y and then integrating first with resect to x.

Solution:

$\frac{216}{35}$

Hint

Sketch the region and follow [link].

Calculating Volumes, Areas, and Average Values

We can use double integrals over general regions to compute volumes, areas, and average values. The methods are the same as those in Double Integrals over Rectangular Regions, but without the restriction to a rectangular region, we can now solve a wider variety of problems.

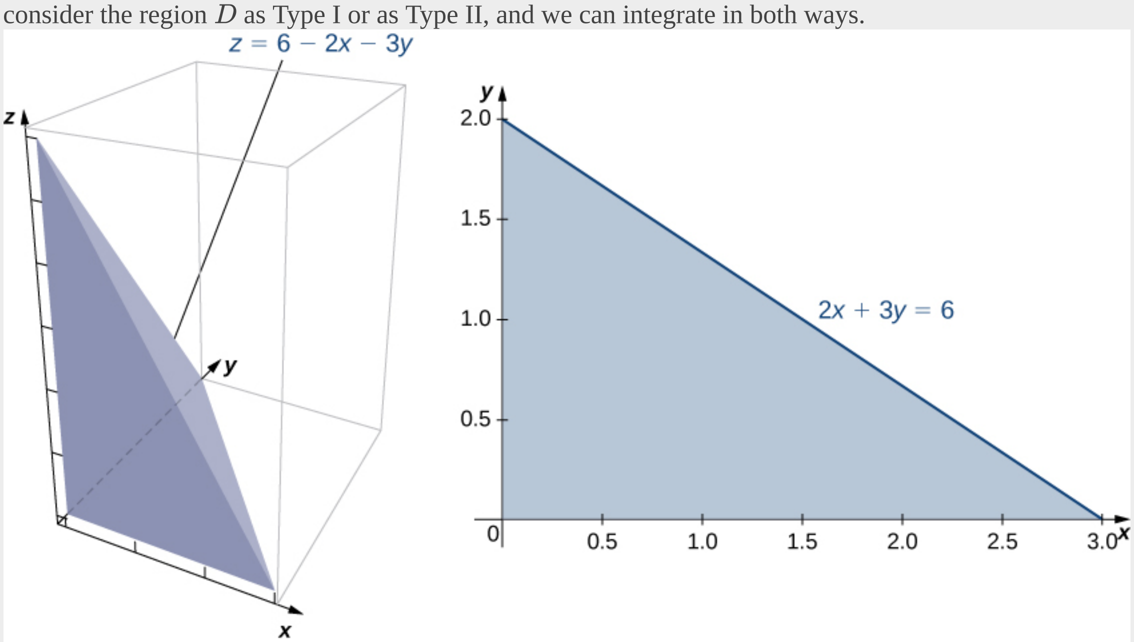
Example:
Exercise:

Problem:
Finding the Volume of a Tetrahedron

Find the volume of the solid bounded by the planes $x = 0, y = 0, z = 0$, and $2x + 3y + z = 6$.

Solution:

The solid is a tetrahedron with the base on the xy-plane and a height $z = 6 - 2x - 3y$. The base is the region D bounded by the lines, $x = 0, y = 0$ and $2x + 3y = 6$ where $z = 0$ ([link]). Note that we can consider the region D as Type I or as Type II, and we can integrate in both ways.



A tetrahedron consisting of the three coordinate planes and the plane $z = 6 - 2x - 3y$, with the base bound by $x = 0, y = 0$, and $2x + 3y = 6$.

First, consider D as a Type I region, and hence $D = \left\{(x, y) | 0 \le x \le 3, 0 \le y \le 2 - \frac{2}{3}x\right\}$.

Therefore, the volume is
Equation:

$$\begin{aligned} V &= \int_{x=0}^{x=3} \int_{y=0}^{y=2-(2x/3)} (6 - 2x - 3y) dy\, dx = \int_{x=0}^{x=3} \left[\left(6y - 2xy - \frac{3}{2}y^2\right)\Big|_{y=0}^{y=2-(2x/3)}\right] dx \\ &= \int_{x=0}^{x=3} \left[\frac{2}{3}(x-3)^2\right] dx = 6. \end{aligned}$$

Now consider D as a Type II region, so $D = \left\{(x, y) \middle| 0 \le y \le 2, 0 \le x \le 3 - \frac{3}{2}y\right\}$. In this calculation, the volume is
Equation:

$$
\begin{aligned}
V &= \int_{y=0}^{y=2} \int_{x=0}^{x=3-(3y/2)} (6-2x-3y)dx\,dy = \int_{y=0}^{y=2} \left[\left(6x - x^2 - 3xy\right)\Big|_{x=0}^{x=3-(3y/2)}\right] dy \\
&= \int_{y=0}^{y=2} \left[\frac{9}{4}(y-2)^2\right] dy = 6.
\end{aligned}
$$

Therefore, the volume is 6 cubic units.

Note:
Exercise:

Problem:

Find the volume of the solid bounded above by $f(x,y) = 10 - 2x + y$ over the region enclosed by the curves $y = 0$ and $y = e^x$, where x is in the interval $[0, 1]$.

Solution:

$\frac{e^2}{4} + 10e - \frac{49}{4}$ cubic units

Hint

Sketch the region, and describe it as Type I.

Finding the area of a rectangular region is easy, but finding the area of a nonrectangular region is not so easy. As we have seen, we can use double integrals to find a rectangular area. As a matter of fact, this comes in very handy for finding the area of a general nonrectangular region, as stated in the next definition.

Note:
Definition

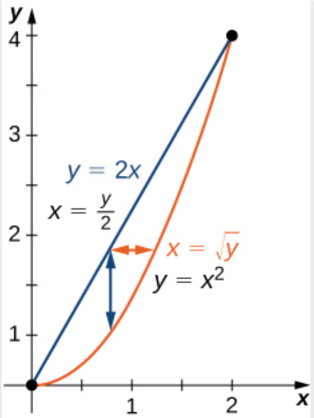
The area of a plane-bounded region D is defined as the double integral $\iint_D 1dA$.

We have already seen how to find areas in terms of single integration. Here we are seeing another way of finding areas by using double integrals, which can be very useful, as we will see in the later sections of this chapter.

Example:
Exercise:

Problem:
Finding the Area of a Region

Find the area of the region bounded below by the curve $y = x^2$ and above by the line $y = 2x$ in the first quadrant ([link]).



The region bounded by $y = x^2$ and $y = 2x$.

Solution:

We just have to integrate the constant function $f(x, y) = 1$ over the region. Thus, the area A of the bounded region is $\int_{x=0}^{x=2} \int_{y=x^2}^{y=2x} dy\, dx$ or $\int_{y=0}^{x=4} \int_{x=y/2}^{x=\sqrt{y}} dx\, dy$:

Equation:

$$A = \iint_D 1 dx\, dy = \int_{x=0}^{x=2} \int_{y=x^2}^{y=2x} 1 dy\, dx = \int_{x=0}^{x=2} \left[y|_{y=x^2}^{y=2x} \right] dx = \int_{x=0}^{x=2} \left(2x - x^2 \right) dx = x^2 - \frac{x^3}{3} \Big|_0^2 = \frac{4}{3}.$$

Note:
Exercise:

Problem:

Find the area of a region bounded above by the curve $y = x^3$ and below by $y = 0$ over the interval $[0, 3]$.

Solution:

$\frac{81}{4}$ square units

Hint

Sketch the region.

We can also use a double integral to find the average value of a function over a general region. The definition is a direct extension of the earlier formula.

Note:
Definition
If $f(x,y)$ is integrable over a plane-bounded region D with positive area $A(D)$, then the average value of the function is
Equation:

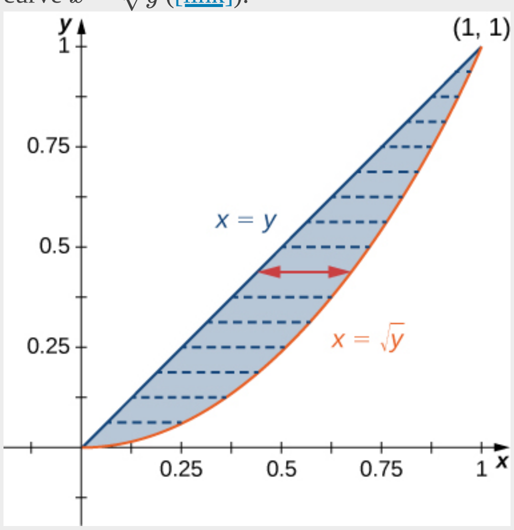
$$f_{ave} = \frac{1}{A(D)} \iint_D f(x,y)dA.$$

Note that the area is $A(D) = \iint_D 1dA$.

Example:
Exercise:

Problem:
Finding an Average Value

Find the average value of the function $f(x,y) = 7xy^2$ on the region bounded by the line $x = y$ and the curve $x = \sqrt{y}$ ([link]).



The region bounded by $x = y$ and $x = \sqrt{y}$.

Solution:

First find the area $A(D)$ where the region D is given by the figure. We have
Equation:

$$A(D) = \iint_D 1dA = \int_{y=0}^{y=1} \int_{x=y}^{x=\sqrt{y}} 1dx\,dy = \int_{y=0}^{y=1} \left[x\Big|_{x=y}^{x=\sqrt{y}} \right] dy = \int_{y=0}^{y=1} (\sqrt{y} - y)dy = \frac{2}{3}y^{3/2} - \frac{y^2}{2}\Big|_0^1 = \frac{1}{6}.$$

Then the average value of the given function over this region is
Equation:

$$\begin{aligned} f_{ave} &= \tfrac{1}{A(D)} \iint\limits_D f(x,y)dA = \frac{1}{A(D)} \int\limits_{y=0}^{y=1} \int\limits_{x=y}^{x=\sqrt{y}} 7xy^2 dx\, dy = \frac{1}{1/6} \int\limits_{y=0}^{y=1} \left[\frac{7}{2} x^2 y^2 \Big|_{x=y}^{x=\sqrt{y}} \right] dy \\ &= 6 \int\limits_{y=0}^{y=1} \left[\frac{7}{2} y^2 \left(y - y^2\right) \right] dy = 6 \int\limits_{y=0}^{y=1} \left[\frac{7}{2} \left(y^3 - y^4\right) \right] dy = \frac{42}{2} \left(\tfrac{y^4}{4} - \tfrac{y^5}{5} \right) \Big|_0^1 = \tfrac{42}{40} = \tfrac{21}{20}. \end{aligned}$$

Note:
Exercise:

Problem:

Find the average value of the function $f(x,y) = xy$ over the triangle with vertices $(0,0)$, $(1,0)$ and $(1,3)$.

Solution:

$\frac{3}{4}$

Hint

Express the line joining $(0,0)$ and $(1,3)$ as a function $y = g(x)$.

Improper Double Integrals

An **improper double integral** is an integral $\iint\limits_D f\, dA$ where either D is an unbounded region or f is an unbounded function. For example, $D = \{(x,y) |\, |x-y| \geq 2\}$ is an unbounded region, and the function $f(x,y) = 1/\left(1 - x^2 - 2y^2\right)$ over the ellipse $x^2 + 3y^2 \leq 1$ is an unbounded function. Hence, both of the following integrals are improper integrals:

i. $\iint\limits_D xy\, dA$ where $D = \{(x,y) |\, |x-y| \geq 2\}$;
ii. $\iint\limits_D \frac{1}{1 - x^2 - 2y^2} dA$ where $D = \left\{(x,y) | x^2 + 3y^2 \leq 1\right\}$.

In this section we would like to deal with improper integrals of functions over rectangles or simple regions such that f has only finitely many discontinuities. Not all such improper integrals can be evaluated; however, a form of Fubini's theorem does apply for some types of improper integrals.

Note:
Fubini's Theorem for Improper Integrals
If D is a bounded rectangle or simple region in the plane defined by $\{(x,y) \colon a \leq x \leq b, g(x) \leq y \leq h(x)\}$ and also by $\{(x,y) \colon c \leq y \leq d, j(y) \leq x \leq k(y)\}$ and f is a nonnegative function on D with finitely many discontinuities in the interior of D, then

Equation:

$$\iint_D f\,dA = \int_{x=a}^{x=b} \int_{y=g(x)}^{y=h(x)} f(x, y)dy\,dx = \int_{y=c}^{y=d} \int_{x=j(y)}^{x=k(y)} f(x, y)dx\,dy.$$

It is very important to note that we required that the function be nonnegative on D for the theorem to work. We consider only the case where the function has finitely many discontinuities inside D.

Example:
Exercise:

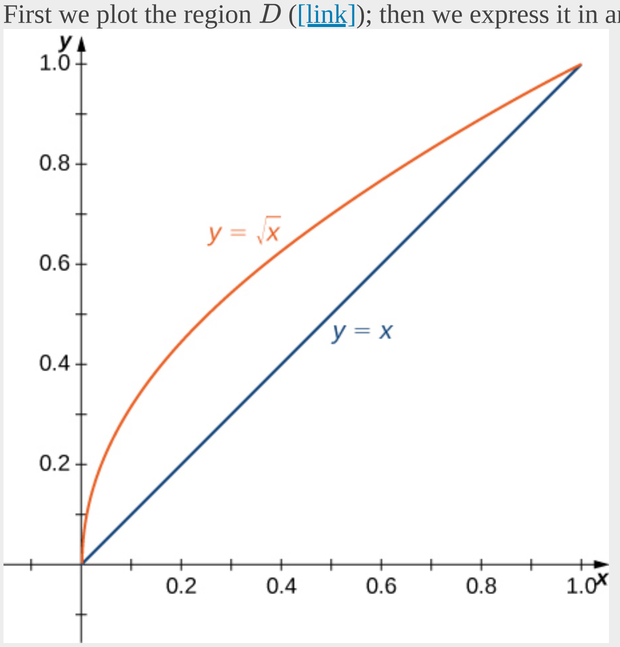
Problem:
Evaluating a Double Improper Integral

Consider the function $f(x, y) = \frac{e^y}{y}$ over the region $D = \{(x, y) \colon 0 \le x \le 1, x \le y \le \sqrt{x}\}$.

Notice that the function is nonnegative and continuous at all points on D except $(0, 0)$. Use Fubini's theorem to evaluate the improper integral.

Solution:

First we plot the region D ([link]); then we express it in another way.



The function f is continuous at all points of the region D except $(0, 0)$.

The other way to express the same region D is
Equation:

$$D = \left\{(x,y): 0 \le y \le 1, y^2 \le x \le y\right\}.$$

Thus we can use Fubini’s theorem for improper integrals and evaluate the integral as

Equation:

$$\int_{y=0}^{y=1} \int_{x=y^2}^{x=y} \frac{e^y}{y} dx\, dy.$$

Therefore, we have

Equation:

$$\int_{y=0}^{y=1} \int_{x=y^2}^{x=y} \frac{e^y}{y} dx\, dy = \int_{y=0}^{y=1} \frac{e^y}{y} x|_{x=y^2}^{x=y} dy = \int_{y=0}^{y=1} \frac{ey}{y}\left(y - y^2\right) dy = \int_0^1 (ey - ye^y) dy = e - 2.$$

As mentioned before, we also have an improper integral if the region of integration is unbounded. Suppose now that the function f is continuous in an unbounded rectangle R.

Note:

Improper Integrals on an Unbounded Region

If R is an unbounded rectangle such as $R = \{(x,y): a \le x \le \infty, c \le y \le \infty\}$, then when the limit exists, we have $\iint_R f(x,y) dA = \lim_{(b,d)\to(\infty,\infty)} \int_a^b \left(\int_c^d f(x,y) dy\right) dx = \lim_{(b,d)\to(\infty,\infty)} \int_c^d \left(\int_a^b f(x,y) dy\right) dy.$

The following example shows how this theorem can be used in certain cases of improper integrals.

Example:

Exercise:

Problem:

Evaluating a Double Improper Integral

Evaluate the integral $\iint_R xye^{-x^2-y^2} dA$ where R is the first quadrant of the plane.

Solution:

The region R is the first quadrant of the plane, which is unbounded. So

Equation:

$$\begin{aligned}\iint\limits_R xye^{-x^2-y^2}dA &= \lim_{(b,d)\to(\infty,\infty)} \int_{x=0}^{x=b} \left(\int_{y=0}^{y=d} xye^{-x^2-y^2}dy \right) dx = \lim_{(b,d)\to(\infty,\infty)} \int_{y=0}^{y=d} \left(\int_{x=0}^{x=b} xye^{-x^2-y^2}dy \right) dy \\ &= \lim_{(b,d)\to(\infty,\infty)} \frac{1}{4}\left(1-e^{-b^2}\right)\left(1-e^{-d^2}\right) = \frac{1}{4}\end{aligned}$$

Thus, $\iint\limits_R xye^{-x^2-y^2}dA$ is convergent and the value is $\frac{1}{4}$.

Note:
Exercise:

Problem:

Evaluate the improper integral $\iint\limits_D \frac{y}{\sqrt{1-x^2-y^2}}dA$ where $D = \left\{(x,y)x \geq 0, y \geq 0, x^2+y^2 \leq 1\right\}$.

Solution:

$\frac{\pi}{4}$

Hint

Notice that the integral is nonnegative and discontinuous on $x^2+y^2=1$. Express the region D as $D = \left\{(x,y)\colon 0 \leq x \leq 1, 0 \leq y \leq \sqrt{1-x^2}\right\}$ and integrate using the method of substitution.

In some situations in probability theory, we can gain insight into a problem when we are able to use double integrals over general regions. Before we go over an example with a double integral, we need to set a few definitions and become familiar with some important properties.

Note:
Definition
Consider a pair of continuous random variables X and Y, such as the birthdays of two people or the number of sunny and rainy days in a month. The joint density function f of X and Y satisfies the probability that (X,Y) lies in a certain region D:
Equation:

$$P\left((X,Y) \in D\right) = \iint\limits_D f\left(x,y\right)dA.$$

Since the probabilities can never be negative and must lie between 0 and 1, the joint density function satisfies the following inequality and equation:
Equation:

$$f\left(x,y\right) \geq 0 \text{ and } \iint\limits_{R^2} f\left(x,y\right)dA = 1.$$

Note:
Definition
The variables X and Y are said to be independent random variables if their joint density function is the product of their individual density functions:
Equation:

$$f\left(x,y\right)=f_1\left(x\right)f_2\left(y\right).$$

Example:
Exercise:

Problem:
Application to Probability

At Sydney's Restaurant, customers must wait an average of 15 minutes for a table. From the time they are seated until they have finished their meal requires an additional 40 minutes, on average. What is the probability that a customer spends less than an hour and a half at the diner, assuming that waiting for a table and completing the meal are independent events?

Solution:

Waiting times are mathematically modeled by exponential density functions, with m being the average waiting time, as
Equation:

$$f(t)=\begin{cases}0 & \text{if } t<0,\\ \frac{1}{m}e^{-t/m} & \text{if } t\geq 0.\end{cases}$$

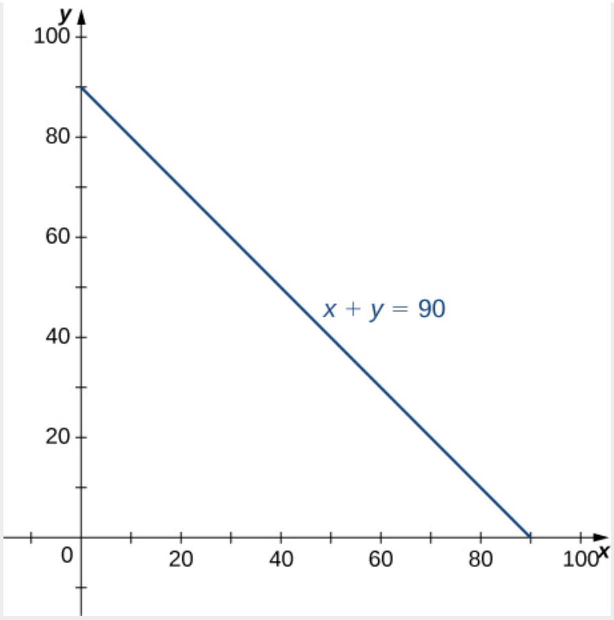
If X and Y are random variables for 'waiting for a table' and 'completing the meal,' then the probability density functions are, respectively,
Equation:

$$f_1(x)=\begin{cases}0 & \text{if } x<0,\\ \frac{1}{15}e^{-x/15} & \text{if } x\geq 0.\end{cases}\text{ and } f_2(y)=\begin{cases}0 & \text{if } y<0,\\ \frac{1}{40}e^{-y/40} & \text{if } y\geq 0.\end{cases}$$

Clearly, the events are independent and hence the joint density function is the product of the individual functions
Equation:

$$f(x,y)=f_1(x)f_2(y)=\begin{cases}0 & \text{if } x<0 \text{ or } y<0,\\ \frac{1}{600}e^{-x/15}e^{-y/60} & \text{if } x,y\geq 0.\end{cases}$$

We want to find the probability that the combined time $X+Y$ is less than 90 minutes. In terms of geometry, it means that the region D is in the first quadrant bounded by the line $x+y=90$ ([link]).



The region of integration for a joint probability density function.

Hence, the probability that (X,Y) is in the region D is
Equation:

$$P(X+Y\le 90)=P((X,Y)\in D)=\iint_D f(x,y)dA=\iint_D \frac{1}{600}e^{-x/15}e^{-y/40}dA.$$

Since $x+y=90$ is the same as $y=90-x$, we have a region of Type I, so
Equation:

$$\begin{aligned} D &= \{(x,y)|0\le x\le 90, 0\le y\le 90-x\}, \\ P(X+Y\le 90) &= \frac{1}{600}\int_{x=0}^{x=90}\int_{y=0}^{y=90-x} e^{-x/15}e^{-y/40}dx\,dy=\frac{1}{600}\int_{x=0}^{x=90}\int_{y=0}^{y=90-x} e^{-x/15}e^{-y/40}dx\,dy \\ &= \frac{1}{600}\int_{x=0}^{x=90}\int_{y=0}^{y=90-x} e^{-(x/15+y/40)}dx\,dy=0.8328. \end{aligned}$$

Thus, there is an 83.2% chance that a customer spends less than an hour and a half at the restaurant.

Another important application in probability that can involve improper double integrals is the calculation of expected values. First we define this concept and then show an example of a calculation.

Note:
Definition
In probability theory, we denote the expected values $E(X)$ and $E(Y)$, respectively, as the most likely outcomes of the events. The expected values $E(X)$ and $E(Y)$ are given by
Equation:

$$E(X) = \iint\limits_{S} x f(x,y) dA \text{ and } E(Y) = \iint\limits_{S} y f(x,y) dA,$$

where S is the sample space of the random variables X and Y.

Example:
Exercise:

Problem:
Finding Expected Value

Find the expected time for the events 'waiting for a table' and 'completing the meal' in [link].

Solution:

Using the first quadrant of the rectangular coordinate plane as the sample space, we have improper integrals for $E(X)$ and $E(Y)$. The expected time for a table is
Equation:

$$\begin{aligned} E(X) &= \iint\limits_{S} x \frac{1}{600} e^{-x/15} e^{-y/40} dA = \frac{1}{600} \int\limits_{x=0}^{x=\infty} \int\limits_{y=0}^{y=\infty} x e^{-x/15} e^{-y/40} dA \\ &= \frac{1}{600} \lim_{(a,b)\to(\infty,\infty)} \int\limits_{x=0}^{x=a} \int\limits_{y=0}^{y=b} x e^{-x/15} e^{-y/40} dx\, dy \\ &= \frac{1}{600} \left(\lim_{a\to\infty} \int\limits_{x=0}^{x=a} x e^{-x/15} dx \right) \left(\lim_{b\to\infty} \int\limits_{y=0}^{y=b} e^{-y/40} dy \right) \\ &= \frac{1}{600} \left(\left(\lim_{a\to\infty} \left(-15 e^{-x/15} (x+15) \right) \right) \Big|_{x=0}^{x=a} \right) \left(\left(\lim_{b\to\infty} \left(-40 e^{-y/40} \right) \right) \Big|_{y=0}^{y=b} \right) \\ &= \frac{1}{600} \left(\lim_{a\to\infty} \left(-15 e^{-a/15} (x+15) + 225 \right) \right) \left(\lim_{b\to\infty} \left(-40 e^{-b/40} + 40 \right) \right) \\ &= \frac{1}{600} (225)(40) \\ &= 15. \end{aligned}$$

A similar calculation shows that $E(Y) = 40$. This means that the expected values of the two random events are the average waiting time and the average dining time, respectively.

Note:
Exercise:

Problem: The joint density function for two random variables X and Y is given by
Equation:

$$f(x,y)=\begin{cases}\frac{1}{600}\left(x^2+y^2\right) & \text{if } 0\le x\le 15, 0\le y\le 10\\ 0 & \text{otherwise}\end{cases}$$

Find the probability that X is at most 10 and Y is at least 5.

Solution:

$\frac{55}{72}\approx 0.7638$

Hint

Compute the probability $P\left(X\le 10, Y\ge 5\right)=\int_{x=-\infty}^{10}\int_{y=5}^{y=10}\frac{1}{6000}\left(x^2+y^2\right)dy\,dx.$

Key Concepts

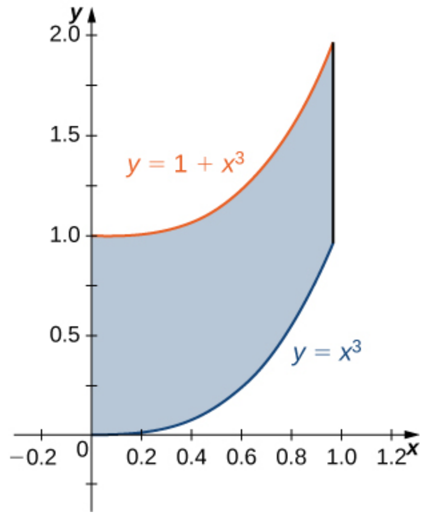
- A general bounded region D on the plane is a region that can be enclosed inside a rectangular region. We can use this idea to define a double integral over a general bounded region.
- To evaluate an iterated integral of a function over a general nonrectangular region, we sketch the region and express it as a Type I or as a Type II region or as a union of several Type I or Type II regions that overlap only on their boundaries.
- We can use double integrals to find volumes, areas, and average values of a function over general regions, similarly to calculations over rectangular regions.
- We can use Fubini's theorem for improper integrals to evaluate some types of improper integrals.

Key Equations

- **Iterated integral over a Type I region**
$$\iint_D f(x,y)dA=\iint_D f(x,y)dy\,dx=\int_a^b\left[\int_{g_1(x)}^{g_2(x)} f(x,y)dy\right]dx$$
- **Iterated integral over a Type II region**
$$\iint_D f(x,y)dA=\iint_D f(x,y)dx\,dy=\int_c^d\left[\int_{h_1(y)}^{h_2(y)} f(x,y)dx\right]dy$$

In the following exercises, specify whether the region is of Type I or Type II.
Exercise:

Problem: The region D bounded by $y=x^3$, $y=x^3+1$, $x=0$, and $x=1$ as given in the following figure.



Exercise:

Problem:

Find the average value of the function $f(x, y) = 3xy$ on the region graphed in the previous exercise.

Solution:

$\frac{27}{20}$

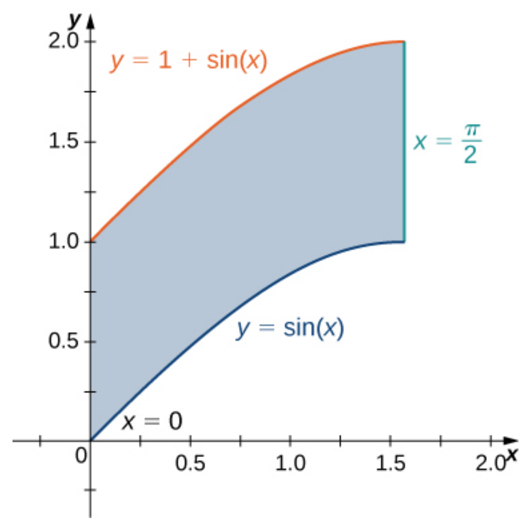
Exercise:

Problem: Find the area of the region D given in the previous exercise.

Exercise:

Problem:

The region D bounded by $y = \sin x, y = 1 + \sin x, x = 0,$ and $x = \frac{\pi}{2}$ as given in the following figure.



Solution:

Type I but not Type II

Exercise:

Problem:

Find the average value of the function $f(x,y) = \cos x$ on the region graphed in the previous exercise.

Exercise:

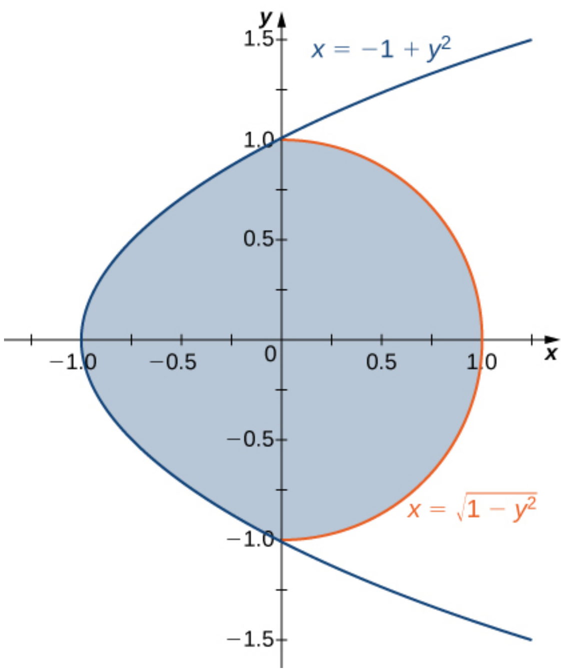
Problem: Find the area of the region D given in the previous exercise.

Solution:

$\frac{\pi}{2}$

Exercise:

Problem: The region D bounded by $x = y^2 - 1$ and $x = \sqrt{1 - y^2}$ as given in the following figure.



Exercise:

Problem:

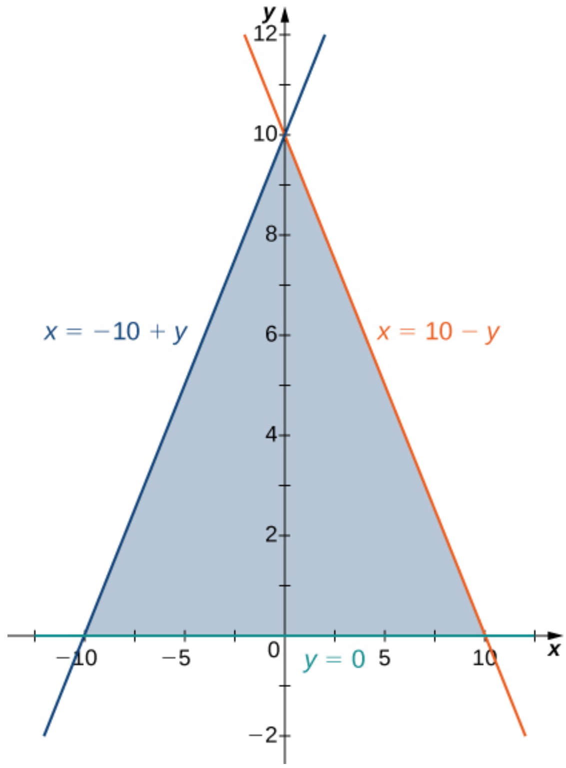
Find the volume of the solid under the graph of the function $f(x,y) = xy + 1$ and above the region in the figure in the previous exercise.

Solution:

$\frac{1}{6}(8 + 3\pi)$

Exercise:

Problem: The region D bounded by $y = 0, x = -10 + y,$ and $x = 10 - y$ as given in the following figure.



Exercise:

Problem:

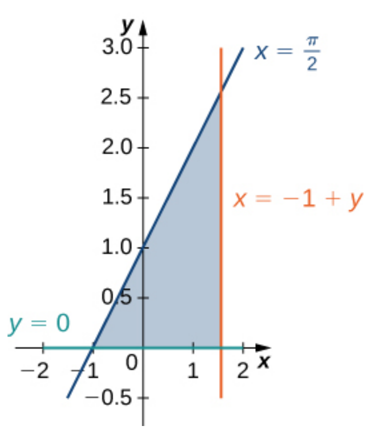
Find the volume of the solid under the graph of the function $f(x, y) = x + y$ and above the region in the figure from the previous exercise.

Solution:

$\frac{1000}{3}$

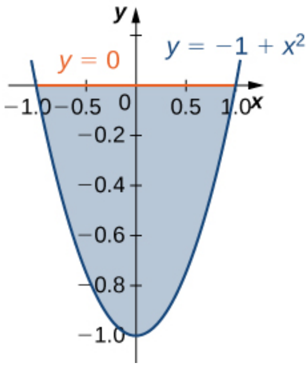
Exercise:

Problem: The region D bounded by $y = 0, x = y - 1, x = \frac{\pi}{2}$ as given in the following figure.



Exercise:

Problem: The region D bounded by $y = 0$ and $y = x^2 - 1$ as given in the following figure.



Solution:

Type I and Type II

Exercise:

Problem:

Let D be the region bounded by the curves of equations $y = x, y = -x$, and $y = 2 - x^2$. Explain why D is neither of Type I nor II.

Exercise:

Problem:

Let D be the region bounded by the curves of equations $y = \cos x$ and $y = 4 - x^2$ and the x-axis. Explain why D is neither of Type I nor II.

Solution:

The region D is not of Type I: it does not lie between two vertical lines and the graphs of two continuous functions $g_1(x)$ and $g_2(x)$. The region D is not of Type II: it does not lie between two horizontal lines and the graphs of two continuous functions $h_1(y)$ and $h_2(y)$.

In the following exercises, evaluate the double integral $\iint\limits_D f(x, y) dA$ over the region D.

Exercise:

Problem: $f(x, y) = 2x + 5y$ and $D = \left\{(x, y)|0 \le x \le 1, x^3 \le y \le x^3 + 1\right\}$

Exercise:

Problem: $f(x, y) = 1$ and $D = \left\{(x, y)|0 \le x \le \frac{\pi}{2}, \sin x \le y \le 1 + \sin x\right\}$

Solution:

$\frac{\pi}{2}$

Exercise:

Problem: $f(x, y) = 2$ and $D = \{(x, y)|0 \le y \le 1, y - 1 \le x \le \arccos y\}$

Exercise:

Problem: $f(x, y) = xy$ and $D = \left\{(x, y)| -1 \le y \le 1, y^2 - 1 \le x \le \sqrt{1 - y^2}\right\}$

Solution:

0

Exercise:

Problem: $f(x, y) = \sin y$ and D is the triangular region with vertices $(0, 0), (0, 3), \text{ and } (3, 0)$

Exercise:

Problem: $f(x, y) = -x + 1$ and D is the triangular region with vertices $(0, 0), (0, 2), \text{ and } (2, 2)$

Solution:

$\frac{2}{3}$

Evaluate the iterated integrals.

Exercise:

Problem: $\int\limits_0^1 \int\limits_{2x}^{3x} \left(x + y^2\right) dy\, dx$

Exercise:

Problem: $\int\limits_0^1 \int\limits_{2\sqrt{x}}^{2\sqrt{x}+1} (xy + 1) dy\, dx$

Solution:

$\frac{41}{20}$

Exercise:

Problem: $\int\limits_e^{e^2} \int\limits_{\ln u}^{2} (v + \ln u) dv\, du$

Exercise:

Problem: $\int\limits_1^2 \int\limits_{-u^2-1}^{-u} (8uv) dv\, du$

Solution:

-63

Exercise:

Problem: $\displaystyle\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \left(2x+4x^3\right) dx\, dy$

Exercise:

Problem: $\displaystyle\int_0^{1/2} \int_{-\sqrt{1-4y^2}}^{\sqrt{1-4y^2}} 4dx\, dy$

Solution:

π

Exercise:

Problem: Let D be the region bounded by $y = 1 - x^2$, $y = 4 - x^2$, and the x- and y-axes.

a. Show that $\displaystyle\iint_D x\, dA = \int_0^1 \int_{1-x^2}^{4-x^2} x\, dy\, dx + \int_1^2 \int_0^{4-x^2} x\, dy\, dx$ by dividing the region D into two regions of Type I.

b. Evaluate the integral $\displaystyle\iint_D x\, dA$.

Exercise:

Problem: Let D be the region bounded by $y = 1$, $y = x$, $y = \ln x$, and the x-axis.

a. Show that $\displaystyle\iint_D y\, dA = \int_0^1 \int_0^x y\, dy\, dx + \int_1^e \int_{\ln x}^1 y\, dy\, dx$ by dividing D into two regions of Type I.

b. Evaluate the integral $\displaystyle\iint_D y\, dA$.

Solution:

a. Answers may vary; b. $\frac{2}{3}$

Exercise:

Problem:

a. Show that $\displaystyle\iint_D y^2 dA = \int_{-1}^0 \int_{-x}^{2-x^2} y^2 dy\, dx + \int_0^1 \int_x^{2-x^2} y^2 dy\, dx$ by dividing the region D into two regions of Type I, where $D = \left\{(x, y) \middle| y \ge x, y \ge -x, y \le 2 - x^2\right\}$.

b. Evaluate the integral $\iint\limits_D y^2 dA$.

Exercise:

Problem: Let D be the region bounded by $y = x^2, y = x + 2$, and $y = -x$.

a. Show that $\iint\limits_D x\,dA = \int_0^1 \int_{-y}^{\sqrt{y}} x\,dx\,dy + \int_1^2 \int_{y-2}^{\sqrt{y}} x\,dx\,dy$ by dividing the region D into two regions of Type II, where $D = \{(x, y) | y \geq x^2, y \geq -x, y \leq x + 2\}$.

b. Evaluate the integral $\iint\limits_D x\,dA$.

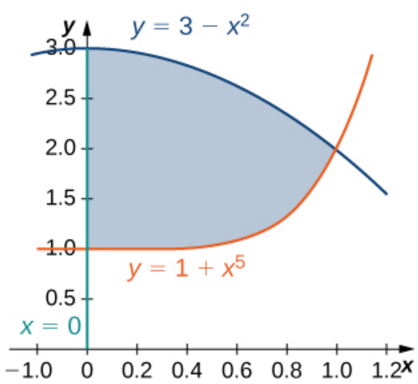
Solution:

a. Answers may vary; b. $\frac{8}{12}$

Exercise:

Problem:

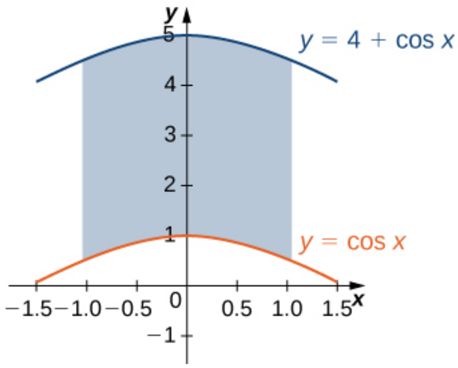
The region D bounded by $x = 0, y = x^5 + 1$, and $y = 3 - x^2$ is shown in the following figure. Find the area $A(D)$ of the region D.



Exercise:

Problem:

The region D bounded by $y = \cos x, y = 4\cos x$, and $x = \pm\frac{\pi}{3}$ is shown in the following figure. Find the area $A(D)$ of the region D.



Solution:

$\frac{8\pi}{3}$

Exercise:

Problem: Find the area $A(D)$ of the region $D = \{(x, y) | y \geq 1 - x^2, y \leq 4 - x^2, y \geq 0, x \geq 0\}$.

Exercise:

Problem:

Let D be the region bounded by $y = 1, y = x, y = \ln x$, and the x-axis. Find the area $A\left(D\right)$ of the region D.

Solution:

$e - \frac{3}{2}$

Exercise:

Problem:

Find the average value of the function $f\left(x, y\right) = \sin y$ on the triangular region with vertices $\left(0, 0\right)$, $\left(0, 3\right)$, and $\left(3, 0\right)$.

Exercise:

Problem:

Find the average value of the function $f\left(x, y\right) = -x + 1$ on the triangular region with vertices $\left(0, 0\right)$, $\left(0, 2\right)$, and $\left(2, 2\right)$.

Solution:

$\frac{2}{3}$

In the following exercises, change the order of integration and evaluate the integral.

Exercise:

Problem: $\int_{-1}^{\pi/2} \int_{0}^{x+1} \sin x \, dy \, dx$

Exercise:

Problem: $\int_{0}^{1} \int_{x-1}^{1-x} x\, dy\, dx$

Solution:

$$\int_{0}^{1} \int_{x-1}^{1-x} x\, dy\, dx = \int_{-1}^{0} \int_{0}^{y+1} x\, dx\, dy + \int_{0}^{1} \int_{0}^{1-y} x\, dx dy = \frac{1}{3}$$

Exercise:

Problem: $\int_{-1}^{0} \int_{-\sqrt{y+1}}^{\sqrt{y+1}} y^2 dx\, dy$

Exercise:

Problem: $\int_{-1/2}^{1/2} \int_{-\sqrt{y^2+1}}^{\sqrt{y^2+1}} y\, dx\, dy$

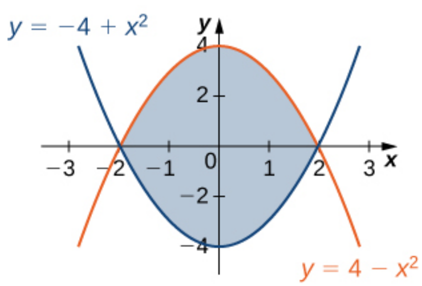
Solution:

$$\int_{-1/2}^{1/2} \int_{-\sqrt{y^2+1}}^{\sqrt{y^2+1}} y\, dx\, dy = \int_{1}^{2} \int_{-\sqrt{x^2-1}}^{\sqrt{x^2-1}} y\, dy\, dx = 0$$

Exercise:

Problem:

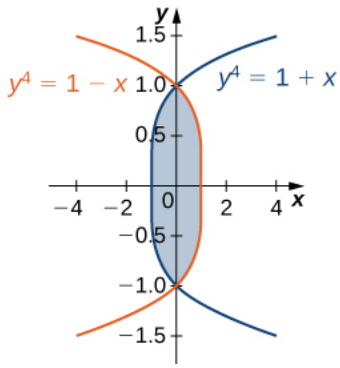
The region D is shown in the following figure. Evaluate the double integral $\iint_D (x^2 + y) dA$ by using the easier order of integration.



Exercise:

Problem:

The region D is given in the following figure. Evaluate the double integral $\iint\limits_D (x^2 - y^2)\,dA$ by using the easier order of integration.



Solution:

$$\iint\limits_D (x^2 - y^2)\,dA = \int_{-1}^{1} \int_{y^4-1}^{1-y^4} (x^2 - y^2)\,dx\,dy = \frac{464}{4095}$$

Exercise:

Problem:

Find the volume of the solid under the surface $z = 2x + y^2$ and above the region bounded by $y = x^5$ and $y = x$.

Exercise:

Problem:

Find the volume of the solid under the plane $z = 3x + y$ and above the region determined by $y = x^7$ and $y = x$.

Solution:

$\frac{4}{5}$

Exercise:

Problem:

Find the volume of the solid under the plane $z = x - y$ and above the region bounded by $x = \tan y$, $x = -\tan y$, and $x = 1$.

Exercise:

Problem:

Find the volume of the solid under the surface $z = x^3$ and above the plane region bounded by $x = \sin y$, $x = -\sin y$, and $x = 1$.

Solution:

$\frac{5\pi}{32}$

Exercise:

Problem:

Let g be a positive, increasing, and differentiable function on the interval $[a, b]$. Show that the volume of the solid under the surface $z = g\prime(x)$ and above the region bounded by $y = 0$, $y = g(x)$, $x = a$, and $x = b$ is given by $\frac{1}{2}\left(g^2(b) - g^2(a)\right)$.

Exercise:

Problem:

Let g be a positive, increasing, and differentiable function on the interval $[a, b]$, and let k be a positive real number. Show that the volume of the solid under the surface $z = g\prime(x)$ and above the region bounded by $y = g(x), y = g(x) + k, x = a$, and $x = b$ is given by $k(g(b) - g(a))$.

Exercise:

Problem:

Find the volume of the solid situated in the first octant and determined by the planes $z = 2$, $z = 0, x + y = 1, x = 0,$ and $y = 0$.

Exercise:

Problem:

Find the volume of the solid situated in the first octant and bounded by the planes $x + 2y = 1$, $x = 0, y = 0, z = 4,$ and $z = 0$.

Solution:

1

Exercise:

Problem:

Find the volume of the solid bounded by the planes $x + y = 1, x - y = 1, x = 0, z = 0,$ and $z = 10$.

Exercise:

Problem:

Find the volume of the solid bounded by the planes $x + y = 1, x - y = 1, x + y = -1,$ $x - y = -1, z = 1$ and $z = 0$.

Solution:

2

Exercise:

Problem:

Let S_1 and S_2 be the solids situated in the first octant under the planes $x + y + z = 1$ and $x + y + 2z = 1$, respectively, and let S be the solid situated between $S_1, S_2, x = 0,$ and $y = 0$.

a. Find the volume of the solid S_1.
b. Find the volume of the solid S_2.
c. Find the volume of the solid S by subtracting the volumes of the solids S_1 and S_2.

Exercise:

Problem:

Let S_1 and S_2 be the solids situated in the first octant under the planes $2x + 2y + z = 2$ and $x + y + z = 1$, respectively, and let S be the solid situated between $S_1, S_2, x = 0,$ and $y = 0.$

a. Find the volume of the solid S_1.
b. Find the volume of the solid S_2.
c. Find the volume of the solid S by subtracting the volumes of the solids S_1 and S_2.

Solution:

a. $\frac{1}{3}$; b. $\frac{1}{6}$; c. $\frac{1}{6}$

Exercise:

Problem:

Let S_1 and S_2 be the solids situated in the first octant under the plane $x + y + z = 2$ and under the sphere $x^2 + y^2 + z^2 = 4$, respectively. If the volume of the solid S_2 is $\frac{4\pi}{3}$, determine the volume of the solid S situated between S_1 and S_2 by subtracting the volumes of these solids.

Exercise:

Problem:

Let S_1 and S_2 be the solids situated in the first octant under the plane $x + y + z = 2$ and bounded by the cylinder $x^2 + y^2 = 4$, respectively.

a. Find the volume of the solid S_1.
b. Find the volume of the solid S_2.
c. Find the volume of the solid S situated between S_1 and S_2 by subtracting the volumes of the solids S_1 and S_2.

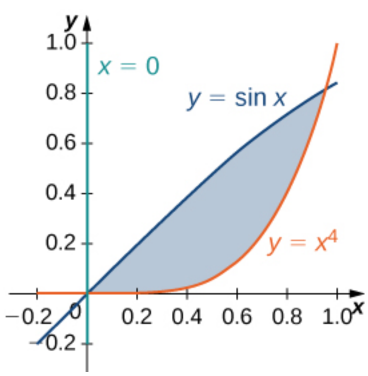
Solution:

a. $\frac{4}{3}$; b. 2π; c. $\frac{6\pi-4}{3}$

Exercise:

Problem:

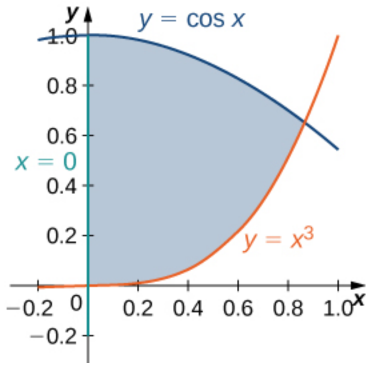
[T] The following figure shows the region D bounded by the curves $y = \sin x$, $x = 0$, and $y = x^4$. Use a graphing calculator or CAS to find the x-coordinates of the intersection points of the curves and to determine the area of the region D. Round your answers to six decimal places.



Exercise:

Problem:

[T] The region D bounded by the curves $y = \cos x, x = 0,$ and $y = x^3$ is shown in the following figure. Use a graphing calculator or CAS to find the *x*-coordinates of the intersection points of the curves and to determine the area of the region D. Round your answers to six decimal places.



Solution:

0 and 0.865474; $A\left(D\right) = 0.621135$

Exercise:

Problem:

Suppose that $\left(X, Y\right)$ is the outcome of an experiment that must occur in a particular region S in the xy-plane. In this context, the region S is called the sample space of the experiment and X and Y are random variables. If D is a region included in S, then the probability of $\left(X, Y\right)$ being in D is defined as $P\left[\left(X, Y\right) \in D\right] = \iint\limits_{D} p\left(x, y\right) dx\, dy$, where $p\left(x, y\right)$ is the joint probability density of the experiment. Here, $p\left(x, y\right)$ is a nonnegative function for which $\iint\limits_{S} p\left(x, y\right) dx\, dy = 1$. Assume that a point $\left(X, Y\right)$ is chosen arbitrarily in the square $\left[0, 3\right] \times \left[0, 3\right]$ with the probability density

$$p\left(x, y\right) = \begin{cases} \frac{1}{9} & \left(x, y\right) \in \left[0, 3\right] \times \left[0, 3\right], \\ 0 & \text{otherwise.} \end{cases}$$

Find the probability that the point $\left(X, Y\right)$ is inside the unit square and interpret the result.

Exercise:

Problem:

Consider X and Y two random variables of probability densities $p_1\left(x\right)$ and $p_2\left(x\right)$, respectively. The random variables X and Y are said to be independent if their joint density function is given by $p\left(x, y\right) = p_1\left(x\right) p_2\left(y\right)$. At a drive-thru restaurant, customers spend, on average, 3 minutes placing their orders and an additional 5 minutes paying for and picking up their meals. Assume that placing the order and paying for/picking up the meal are two independent events X and Y. If the waiting times are modeled by the exponential probability densities

$$p_1(x) = \begin{cases} \frac{1}{3}e^{-x/3} & x \geq 0, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad p_2(y) = \begin{cases} \frac{1}{5}e^{-y/5} & y \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

respectively, the probability that a customer will spend less than 6 minutes in the drive-thru line is given by $P[X + Y \leq 6] = \iint\limits_D p(x, y)dx\,dy$, where $D = \{(x, y)\}|x \geq 0, y \geq 0, x + y \leq 6\}$. Find $P[X + Y \leq 6]$ and interpret the result.

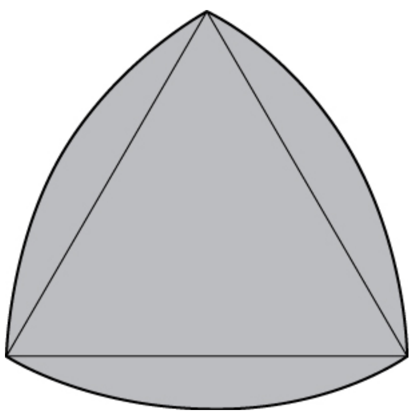
Solution:

$P[X + Y \leq 6] = 1 + \frac{3}{2e^2} - \frac{5}{e^{6/5}} \approx 0.45$; there is a 45% chance that a customer will spend 6 minutes in the drive-thru line.

Exercise:

Problem:

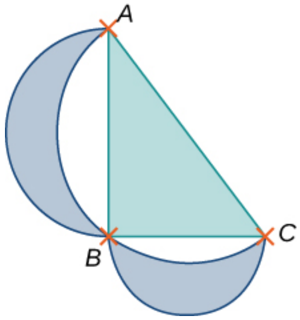
[T] The Reuleaux triangle consists of an equilateral triangle and three regions, each of them bounded by a side of the triangle and an arc of a circle of radius *s* centered at the opposite vertex of the triangle. Show that the area of the Reuleaux triangle in the following figure of side length s is $\frac{s^2}{2}\left(\pi - \sqrt{3}\right)$.



Exercise:

Problem:

[T] Show that the area of the lunes of Alhazen, the two blue lunes in the following figure, is the same as the area of the right triangle *ABC*. The outer boundaries of the lunes are semicircles of diameters AB and AC, respectively, and the inner boundaries are formed by the circumcircle of the triangle ABC.



Glossary

improper double integral
: a double integral over an unbounded region or of an unbounded function

Type I
: a region D in the xy-plane is Type I if it lies between two vertical lines and the graphs of two continuous functions $g_1(x)$ and $g_2(x)$

Type II
: a region D in the xy-plane is Type II if it lies between two horizontal lines and the graphs of two continuous functions $h_1(y)$ and $h_2(y)$

Double Integrals in Polar Coordinates

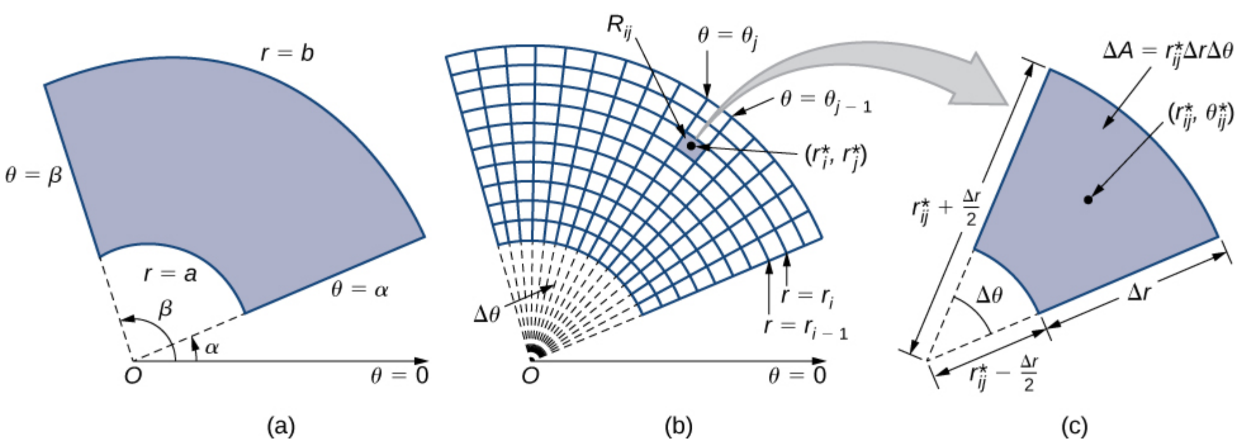
- Recognize the format of a double integral over a polar rectangular region.
- Evaluate a double integral in polar coordinates by using an iterated integral.
- Recognize the format of a double integral over a general polar region.
- Use double integrals in polar coordinates to calculate areas and volumes.

Double integrals are sometimes much easier to evaluate if we change rectangular coordinates to polar coordinates. However, before we describe how to make this change, we need to establish the concept of a double integral in a polar rectangular region.

Polar Rectangular Regions of Integration

When we defined the double integral for a continuous function in rectangular coordinates—say, g over a region R in the xy-plane—we divided R into subrectangles with sides parallel to the coordinate axes. These sides have either constant x-values and/or constant y-values. In polar coordinates, the shape we work with is a **polar rectangle**, whose sides have constant r-values and/or constant θ-values. This means we can describe a polar rectangle as in [link](a), with $R = \{(r, \theta) | a \leq r \leq b, \alpha \leq \theta \leq \beta\}$.

In this section, we are looking to integrate over polar rectangles. Consider a function $f\,(r, \theta)$ over a polar rectangle R. We divide the interval $[a, b]$ into m subintervals $[r_{i-1}, r_i]$ of length $\Delta r = (b - a)/m$ and divide the interval $[\alpha, \beta]$ into n subintervals $[\theta_{i-1}, \theta_i]$ of width $\Delta\theta = (\beta - \alpha)/n$. This means that the circles $r = r_i$ and rays $\theta = \theta_i$ for $1 \leq i \leq m$ and $1 \leq j \leq n$ divide the polar rectangle R into smaller polar subrectangles R_{ij} ([link](b)).



(a) A polar rectangle R (b) divided into subrectangles R_{ij}. (c) Close-up of a subrectangle.

As before, we need to find the area ΔA of the polar subrectangle R_{ij} and the "polar" volume of the thin box above R_{ij}. Recall that, in a circle of radius r, the length s of an arc subtended by a central angle of θ radians is $s = r\theta$. Notice that the polar rectangle R_{ij} looks a lot like a trapezoid with parallel sides $r_{i-1}\Delta\theta$ and $r_i\Delta\theta$ and with a width Δr. Hence the area of the polar subrectangle R_{ij} is

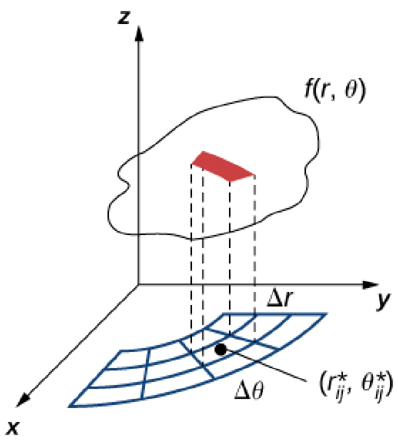
Equation:

$$\Delta A = \frac{1}{2}\Delta r\,(r_{i-1}\Delta\theta + r_1\Delta\theta).$$

Simplifying and letting $\overset{*}{r}_{ij} = \frac{1}{2}(r_{i-1} + r_i)$, we have $\Delta A = \overset{*}{r}_{ij}\Delta r\Delta\theta$. Therefore, the polar volume of the thin box above R_{ij} ([link]) is

Equation:

$$f(\overset{*}{r}_{ij}, \overset{*}{\theta}_{ij})\Delta A = f(\overset{*}{r}_{ij}, \overset{*}{\theta}_{ij})\overset{*}{r}_{ij}\Delta r\Delta\theta.$$



Finding the volume of the thin box above polar rectangle R_{ij}.

Using the same idea for all the subrectangles and summing the volumes of the rectangular boxes, we obtain a double Riemann sum as

Equation:

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(\overset{*}{r}_{ij}, \overset{*}{\theta}_{ij})\overset{*}{r}_{ij}\Delta r\Delta\theta.$$

As we have seen before, we obtain a better approximation to the polar volume of the solid above the region R when we let m and n become larger. Hence, we define the polar volume as the limit of the double Riemann sum,

Equation:

$$V = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(\overset{*}{r}_{ij}, \overset{*}{\theta}_{ij})\overset{*}{r}_{ij}\Delta r\Delta\theta.$$

This becomes the expression for the double integral.

Note:
Definition
The double integral of the function $f(r, \theta)$ over the polar rectangular region R in the $r\theta$-plane is defined as
Equation:

$$\iint_R f(r, \theta) dA = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(r_{ij}^*, \theta_{ij}^*) \Delta A = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(r_{ij}^*, \theta_{ij}^*) r_{ij}^* \Delta r \Delta \theta.$$

Again, just as in Double Integrals over Rectangular Regions, the double integral over a polar rectangular region can be expressed as an iterated integral in polar coordinates. Hence,
Equation:

$$\iint_R f(r, \theta) dA = \iint_R f(r, \theta) r\, dr\, d\theta = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=a}^{r=b} f(r, \theta) r\, dr\, d\theta.$$

Notice that the expression for dA is replaced by $r\, dr\, d\theta$ when working in polar coordinates. Another way to look at the polar double integral is to change the double integral in rectangular coordinates by substitution. When the function f is given in terms of x and y, using $x = r \cos \theta, y = r \sin \theta,$ and $dA = r\, dr\, d\theta$ changes it to
Equation:

$$\iint_R f(x, y) dA = \iint_R f(r \cos \theta, r \sin \theta) r\, dr\, d\theta.$$

Note that all the properties listed in Double Integrals over Rectangular Regions for the double integral in rectangular coordinates hold true for the double integral in polar coordinates as well, so we can use them without hesitation.

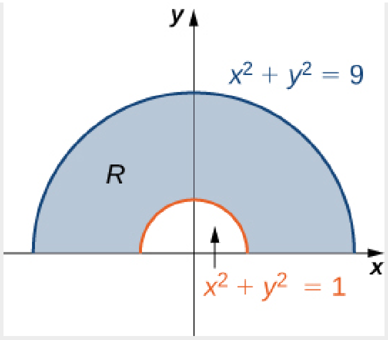
Example:
Exercise:

Problem:
Sketching a Polar Rectangular Region

Sketch the polar rectangular region $R = \{(r, \theta) | 1 \le r \le 3, 0 \le \theta \le \pi\}$.

Solution:

As we can see from [link], $r = 1$ and $r = 3$ are circles of radius 1 and 3 and $0 \le \theta \le \pi$ covers the entire top half of the plane. Hence the region R looks like a semicircular band.



The polar region R lies between two semicircles.

Now that we have sketched a polar rectangular region, let us demonstrate how to evaluate a double integral over this region by using polar coordinates.

Example:
Exercise:

Problem:
Evaluating a Double Integral over a Polar Rectangular Region

Evaluate the integral $\iint\limits_R 3x\,dA$ over the region $R = \{(r,\theta)|1 \le r \le 2, 0 \le \theta \le \pi\}$.

Solution:

First we sketch a figure similar to [link] but with outer radius 2. From the figure we can see that we have

Equation:

$$\begin{aligned} \iint\limits_R 3x\,dA &= \int_{\theta=0}^{\theta=\pi}\int_{r=1}^{r=2} 3r\cos\theta r\,dr\,d\theta && \text{Use an iterated integral with correct limits of integration.} \\ &= \int_{\theta=0}^{\theta=\pi} \cos\theta \left[r^3\Big|_{r=1}^{r=2}\right] d\theta && \text{Integrate first with respect to } r. \\ &= \int_{\theta=0}^{\theta=\pi} 7\cos\theta\,d\theta = 7\sin\theta\Big|_{\theta=0}^{\theta=\pi} = 0. \end{aligned}$$

Note:
Exercise:

Problem: Sketch the region $R = \left\{(r,\theta)|1 \le r \le 2, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right\}$, and evaluate $\iint\limits_R x\,dA$.

Solution:

$\frac{14}{3}$

Hint

Follow the steps in [link].

Example:
Exercise:

Problem:
Evaluating a Double Integral by Converting from Rectangular Coordinates

Evaluate the integral $\iint\limits_R \left(1 - x^2 - y^2\right) dA$ where R is the unit circle on the xy-plane.

Solution:

The region R is a unit circle, so we can describe it as $R = \{(r,\theta)|0 \le r \le 1, 0 \le \theta \le 2\pi\}$.

Using the conversion $x = r\cos\theta, y = r\sin\theta$, and $dA = r\,dr\,d\theta$, we have
Equation:

$$\begin{aligned} \iint\limits_R \left(1 - x^2 - y^2\right) dA &= \int_0^{2\pi}\int_0^1 \left(1 - r^2\right) r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \left(r - r^3\right) dr\,d\theta \\ &= \int_0^{2\pi} \left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1 d\theta = \int_0^{2\pi} \frac{1}{4} d\theta = \frac{\pi}{2}. \end{aligned}$$

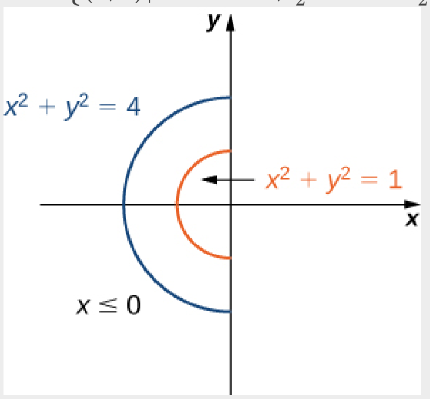
Example:
Exercise:

Problem:
Evaluating a Double Integral by Converting from Rectangular Coordinates

Evaluate the integral $\iint\limits_R (x + y) dA$ where $R = \left\{(x,y)|1 \le x^2 + y^2 \le 4, x \le 0\right\}$.

Solution:

We can see that R is an annular region that can be converted to polar coordinates and described as $R = \left\{(r,\theta)|1 \le r \le 2, \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}\right\}$ (see the following graph).



The annular region of integration R.

Hence, using the conversion $x = r\cos\theta, y = r\sin\theta$, and $dA = r\,dr\,d\theta$, we have

Equation:

$$\begin{aligned}\iint\limits_R (x+y)dA &= \int\limits_{\theta=\pi/2}^{\theta=3\pi/2}\int\limits_{r=1}^{r=2}(r\cos\theta + r\sin\theta)r\,dr\,d\theta \\ &= \left(\int\limits_{r=1}^{r=2} r^2 dr\right)\left(\int\limits_{\pi/2}^{3\pi/2}(\cos\theta+\sin\theta)d\theta\right) \\ &= \left[\frac{r^3}{3}\right]_1^2 [\sin\theta - \cos\theta]\Big|_{\pi/2}^{3\pi/2} \\ &= -\frac{14}{3}.\end{aligned}$$

Note:
Exercise:

Problem:

Evaluate the integral $\iint\limits_R \left(4 - x^2 - y^2\right)dA$ where R is the circle of radius 2 on the xy-plane.

Solution:

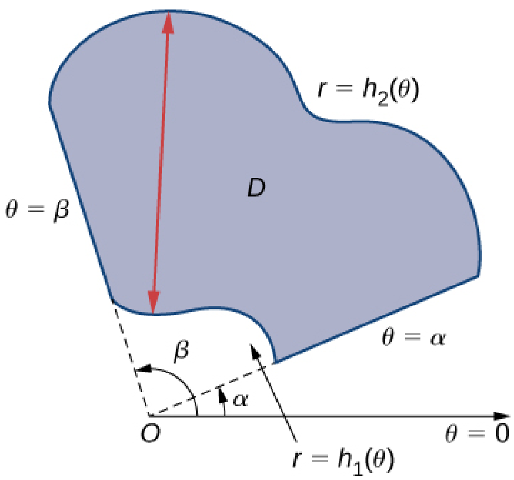
8π

Hint

Follow the steps in the previous example.

General Polar Regions of Integration

To evaluate the double integral of a continuous function by iterated integrals over general polar regions, we consider two types of regions, analogous to Type I and Type II as discussed for rectangular coordinates in Double Integrals over General Regions. It is more common to write polar equations as $r = f(\theta)$ than $\theta = f(r)$, so we describe a general polar region as $R = \{(r, \theta) | \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$ (see the following figure).



A general polar region between $\alpha < \theta < \beta$ and $h_1(\theta) < r < h_2(\theta)$.

Note:
Double Integrals over General Polar Regions
If $f(r, \theta)$ is continuous on a general polar region D as described above, then
Equation:

$$\iint_D f(r,\theta) r \, dr \, d\theta = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} f(r,\theta) r \, dr \, d\theta$$

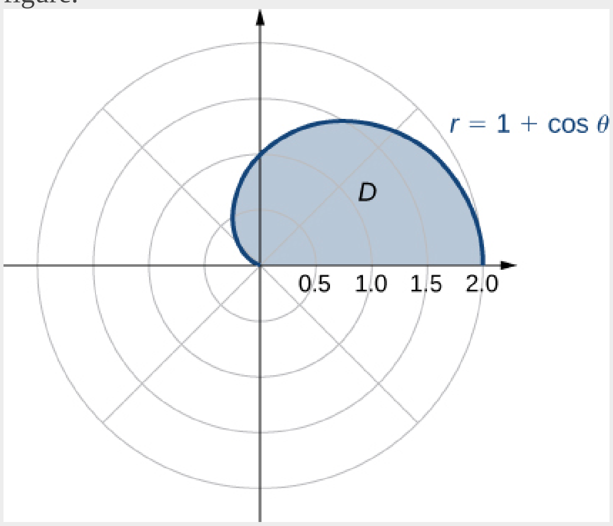
Example:
Exercise:

Problem:
Evaluating a Double Integral over a General Polar Region

Evaluate the integral $\iint_D r^2 \sin\theta r\, dr\, d\theta$ where D is the region bounded by the polar axis and the upper half of the cardioid $r = 1 + \cos\theta$.

Solution:

We can describe the region D as $\{(r, \theta) | 0 \le \theta \le \pi, 0 \le r \le 1 + \cos\theta\}$ as shown in the following figure.



The region D is the top half of a cardioid.

Hence, we have

Equation:

$$\begin{aligned}\iint_D r^2 \sin\theta r\, dr\, d\theta &= \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=1+\cos\theta} \left(r^2 \sin\theta\right) r\, dr\, d\theta \\ &= \frac{1}{4} \int_{\theta=0}^{\theta=\pi} \left[r^4\right]_{r=0}^{r=1+\cos\theta} \sin\theta\, d\theta \\ &= \frac{1}{4} \int_{\theta=0}^{\theta=\pi} (1 + \cos\theta)^4 \sin\theta\, d\theta \\ &= -\frac{1}{4} \left[\frac{(1+\cos\theta)^5}{5}\right]_0^{\pi} = \frac{8}{5}.\end{aligned}$$

Note:
Exercise:

Problem: Evaluate the integral
Equation:

$$\iint_D r^2 \sin^2 2\theta r\, dr\, d\theta \text{ where } D = \left\{(r,\theta)|0 \le \theta \le \pi, 0 \le r \le 2\sqrt{\cos 2\theta}\right\}.$$

Solution:

$\pi/8$

Hint

Graph the region and follow the steps in the previous example.

Polar Areas and Volumes

As in rectangular coordinates, if a solid S is bounded by the surface $z = f(r,\theta)$, as well as by the surfaces $r = a, r = b, \theta = \alpha$, and $\theta = \beta$, we can find the volume V of S by double integration, as
Equation:

$$V = \iint_R f(r,\theta) r\, dr\, d\theta = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=a}^{r=b} f(r,\theta) r\, dr\, d\theta.$$

If the base of the solid can be described as $D = \{(r,\theta)|\alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$, then the double integral for the volume becomes
Equation:

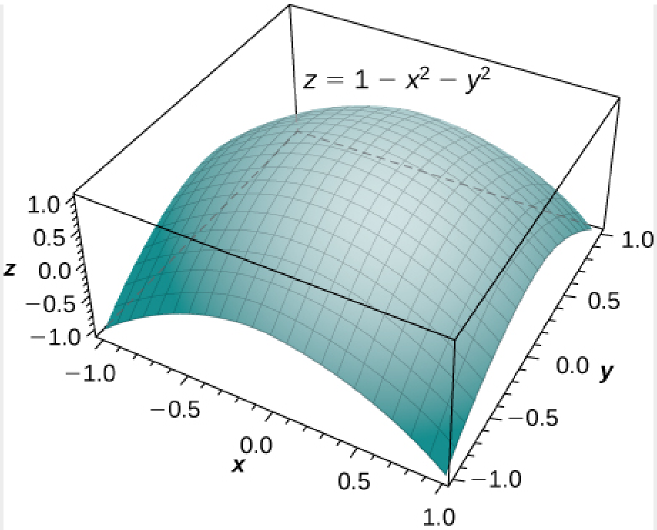
$$V = \iint_D f(r,\theta) r\, dr\, d\theta = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} f(r,\theta) r\, dr\, d\theta.$$

We illustrate this idea with some examples.

Example:
Exercise:

Problem:
Finding a Volume Using a Double Integral

Find the volume of the solid that lies under the paraboloid $z = 1 - x^2 - y^2$ and above the unit circle on the xy-plane (see the following figure).



The paraboloid $z = 1 - x^2 - y^2$.

Solution:

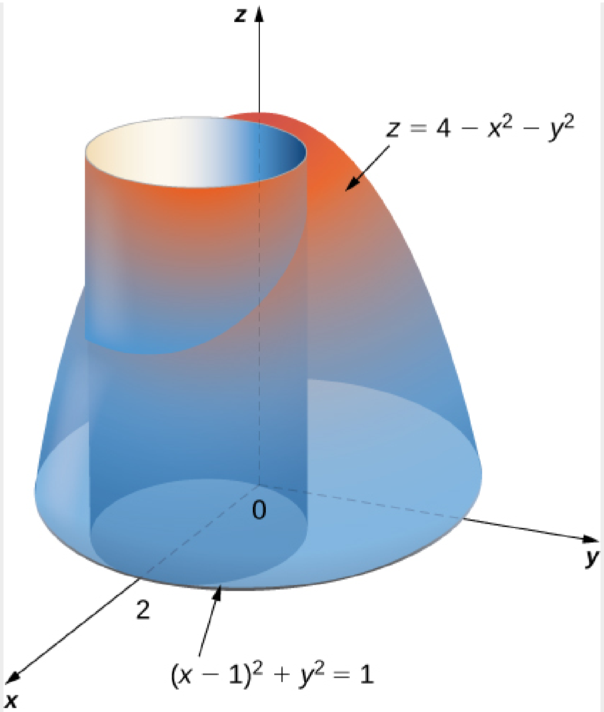
By the method of double integration, we can see that the volume is the iterated integral of the form $\iint_R (1 - x^2 - y^2)\,dA$ where $R = \{(r, \theta) | 0 \le r \le 1, 0 \le \theta \le 2\pi\}$.

This integration was shown before in [link], so the volume is $\frac{\pi}{2}$ cubic units.

Example:
Exercise:

Problem:
Finding a Volume Using Double Integration

Find the volume of the solid that lies under the paraboloid $z = 4 - x^2 - y^2$ and above the disk $(x-1)^2 + y^2 = 1$ on the xy-plane. See the paraboloid in [link] intersecting the cylinder $(x-1)^2 + y^2 = 1$ above the xy-plane.



Finding the volume of a solid with a paraboloid cap and a circular base.

Solution:

First change the disk $(x-1)^2 + y^2 = 1$ to polar coordinates. Expanding the square term, we have $x^2 - 2x + 1 + y^2 = 1$. Then simplify to get $x^2 + y^2 = 2x$, which in polar coordinates becomes $r^2 = 2r \cos \theta$ and then either $r = 0$ or $r = 2 \cos \theta$. Similarly, the equation of the paraboloid changes to $z = 4 - r^2$. Therefore we can describe the disk $(x-1)^2 + y^2 = 1$ on the xy-plane as the region
Equation:

$$D = \{(r, \theta) | 0 \leq \theta \leq \pi, 0 \leq r \leq 2 \cos \theta\}.$$

Hence the volume of the solid bounded above by the paraboloid $z = 4 - x^2 - y^2$ and below by $r = 2 \cos \theta$ is
Equation:

$$\begin{aligned} V &= \iint_D f(r,\theta) r\,dr\,d\theta = \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=2\cos\theta} (4-r^2) r\,dr\,d\theta \\ &= \int_{\theta=0}^{\theta=\pi} \left[4\frac{r^2}{2} - \frac{r^4}{4} \Big|_0^{2\cos\theta} \right] d\theta \\ &= \int_0^{\pi} \left[8\cos^2\theta - 4\cos^2\theta \right] d\theta = \left[\tfrac{5}{2}\theta + \tfrac{5}{2}\sin\theta\cos\theta - \sin\theta\cos^3\theta \right]_0^{\pi} = \tfrac{5}{2}\pi. \end{aligned}$$

Notice in the next example that integration is not always easy with polar coordinates. Complexity of integration depends on the function and also on the region over which we need to perform the integration. If the region has a more natural expression in polar coordinates or if f has a simpler antiderivative in polar coordinates, then the change in polar coordinates is appropriate; otherwise, use rectangular coordinates.

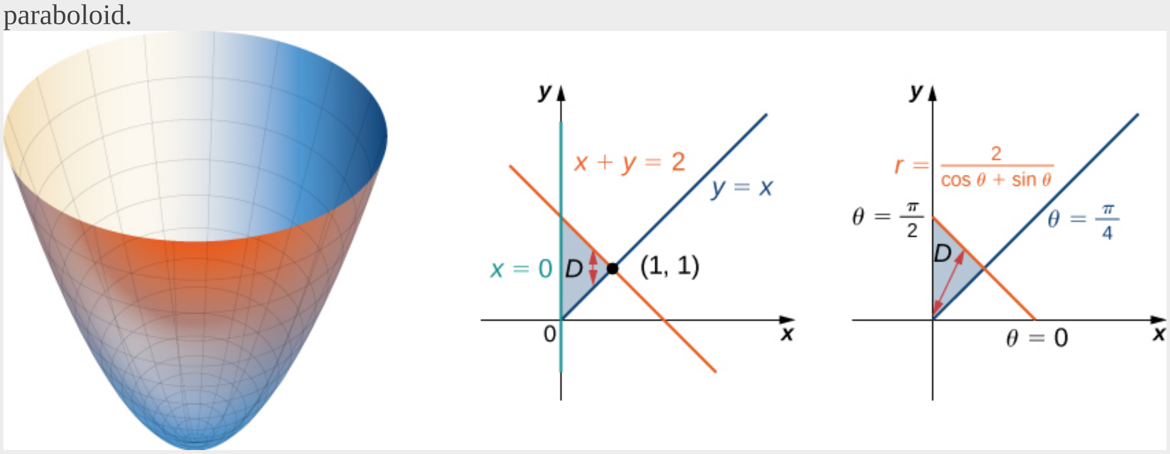
Example:
Exercise:

Problem:
Finding a Volume Using a Double Integral

Find the volume of the region that lies under the paraboloid $z = x^2 + y^2$ and above the triangle enclosed by the lines $y = x$, $x = 0$, and $x + y = 2$ in the xy-plane ([link]).

Solution:

First examine the region over which we need to set up the double integral and the accompanying paraboloid.



Finding the volume of a solid under a paraboloid and above a given triangle.

The region D is $\{(x, y) | 0 \leq x \leq 1, x \leq y \leq 2 - x\}$. Converting the lines $y = x, x = 0$, and $x + y = 2$ in the xy-plane to functions of r and θ, we have $\theta = \pi/4$, $\theta = \pi/2$, and $r = 2/(\cos\theta + \sin\theta)$, respectively. Graphing the region on the xy-plane, we see that it looks like $D = \{(r, \theta) | \pi/4 \leq \theta \leq \pi/2, 0 \leq r \leq 2/(\cos\theta + \sin\theta)\}$. Now converting the equation of the surface gives $z = x^2 + y^2 = r^2$. Therefore, the volume of the solid is given by the double integral

Equation:

$$\begin{aligned} V &= \iint_D f(r, \theta) r\, dr\, d\theta = \int_{\theta=\pi/4}^{\theta=\pi/2} \int_{r=0}^{r=2/(\cos\theta+\sin\theta)} r^2 r\, dr\, d\theta = \int_{\pi/4}^{\pi/2} \left[\frac{r^4}{4}\right]_0^{2/(\cos\theta+\sin\theta)} d\theta \\ &= \frac{1}{4}\int_{\pi/4}^{\pi/2} \left(\frac{2}{\cos\theta + \sin\theta}\right)^4 d\theta = \frac{16}{4}\int_{\pi/4}^{\pi/2} \left(\frac{1}{\cos\theta + \sin\theta}\right)^4 d\theta = 4\int_{\pi/4}^{\pi/2} \left(\frac{1}{\cos\theta + \sin\theta}\right)^4 d\theta. \end{aligned}$$

As you can see, this integral is very complicated. So, we can instead evaluate this double integral in rectangular coordinates as

Equation:

$$V = \int_0^1 \int_x^{2-x} (x^2 + y^2)\, dy\, dx.$$

Evaluating gives

Equation:

$$\begin{aligned} V &= \int_0^1 \int_x^{2-x} (x^2 + y^2)\, dy\, dx = \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]\Bigg|_x^{2-x} dx \\ &= \int_0^1 \frac{8}{3} - 4x + 4x^2 - \frac{8x^3}{3}\, dx \\ &= \left[\frac{8x}{3} - 2x^2 + \frac{4x^3}{3} - \frac{2x^4}{3}\right]\Bigg|_0^1 = \frac{4}{3}. \end{aligned}$$

To answer the question of how the formulas for the volumes of different standard solids such as a sphere, a cone, or a cylinder are found, we want to demonstrate an example and find the volume of an arbitrary cone.

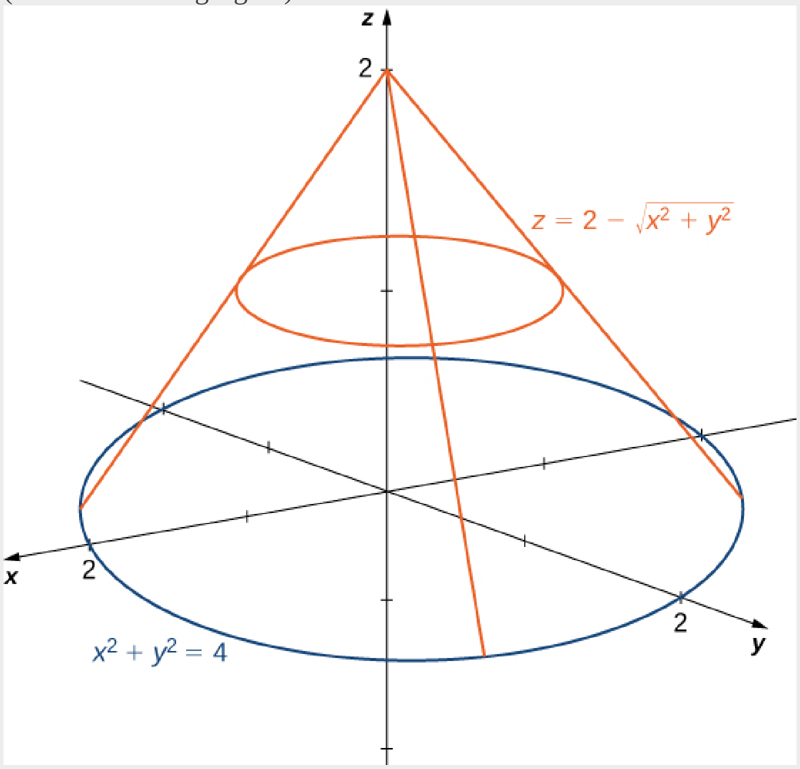
Example:
Exercise:

Problem:
Finding a Volume Using a Double Integral

Use polar coordinates to find the volume inside the cone $z = 2 - \sqrt{x^2 + y^2}$ and above the xy-plane.

Solution:

The region D for the integration is the base of the cone, which appears to be a circle on the xy-plane (see the following figure).



Finding the volume of a solid inside the cone and above the xy-plane.

We find the equation of the circle by setting $z = 0$:

Equation:

$$\begin{aligned} 0 &= 2 - \sqrt{x^2 + y^2} \\ 2 &= \sqrt{x^2 + y^2} \\ x^2 + y^2 &= 4. \end{aligned}$$

This means the radius of the circle is 2, so for the integration we have $0 \le \theta \le 2\pi$ and $0 \le r \le 2$. Substituting $x = r\cos\theta$ and $y = r\sin\theta$ in the equation $z = 2 - \sqrt{x^2 + y^2}$ we have $z = 2 - r$. Therefore, the volume of the cone is

$$\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} (2-r) r \, dr \, d\theta = 2\pi \frac{4}{3} = \frac{8\pi}{3} \text{ cubic units.}$$

Analysis

Note that if we were to find the volume of an arbitrary cone with radius a units and height h units, then the equation of the cone would be $z = h - \frac{h}{a}\sqrt{x^2+y^2}$.

We can still use [link] and set up the integral as $\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=a} \left(h - \frac{h}{a} r\right) r \, dr \, d\theta$.

Evaluating the integral, we get $\frac{1}{3}\pi a^2 h$.

Note:
Exercise:

Problem:

Use polar coordinates to find an iterated integral for finding the volume of the solid enclosed by the paraboloids $z = x^2 + y^2$ and $z = 16 - x^2 - y^2$.

Solution:

$$V = \int_0^{2\pi} \int_0^{2\sqrt{2}} \left(16 - 2r^2\right) r \, dr \, d\theta = 64\pi \text{ cubic units}$$

Hint

Sketching the graphs can help.

As with rectangular coordinates, we can also use polar coordinates to find areas of certain regions using a double integral. As before, we need to understand the region whose area we want to compute. Sketching a graph and identifying the region can be helpful to realize the limits of integration. Generally, the area formula in double integration will look like

Equation:

$$\text{Area } A = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} 1r \, dr \, d\theta.$$

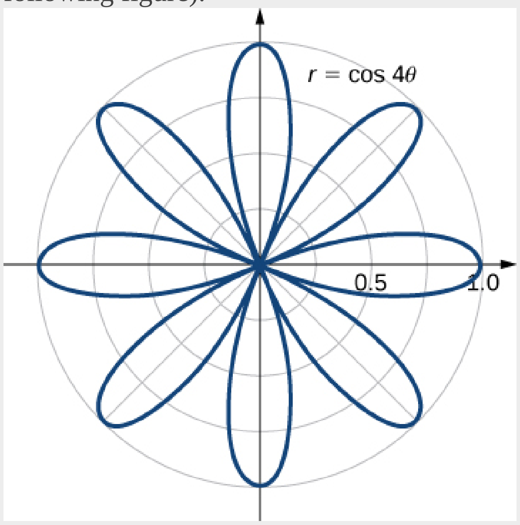
Example:
Exercise:

Problem:
Finding an Area Using a Double Integral in Polar Coordinates

Evaluate the area bounded by the curve $r = \cos 4\theta$.

Solution:

Sketching the graph of the function $r = \cos 4\theta$ reveals that it is a polar rose with eight petals (see the following figure).



Finding the area of a polar rose with eight petals.

Using symmetry, we can see that we need to find the area of one petal and then multiply it by 8. Notice that the values of θ for which the graph passes through the origin are the zeros of the function $\cos 4\theta$, and these are odd multiples of $\pi/8$. Thus, one of the petals corresponds to the values of θ in the interval $[-\pi/8, \pi/8]$. Therefore, the area bounded by the curve $r = \cos 4\theta$ is

Equation:

$$\begin{aligned} A &= 8 \int_{\theta=-\pi/8}^{\theta=\pi/8} \int_{r=0}^{r=\cos 4\theta} 1r\,dr\,d\theta \\ &= 8 \int_{-\pi/8}^{\pi/8} \left[\frac{1}{2} r^2 \Big|_0^{\cos 4\theta}\right] d\theta = 8 \int_{-\pi/8}^{\pi/8} \frac{1}{2} \cos^2 4\theta\,d\theta = 8 \left[\tfrac{1}{4}\theta + \tfrac{1}{16} \sin 4\theta \cos 4\theta \Big|_{-\pi/8}^{\pi/8}\right] = 8 \left[\tfrac{\pi}{16}\right] = \tfrac{\pi}{2}. \end{aligned}$$

Example:
Exercise:

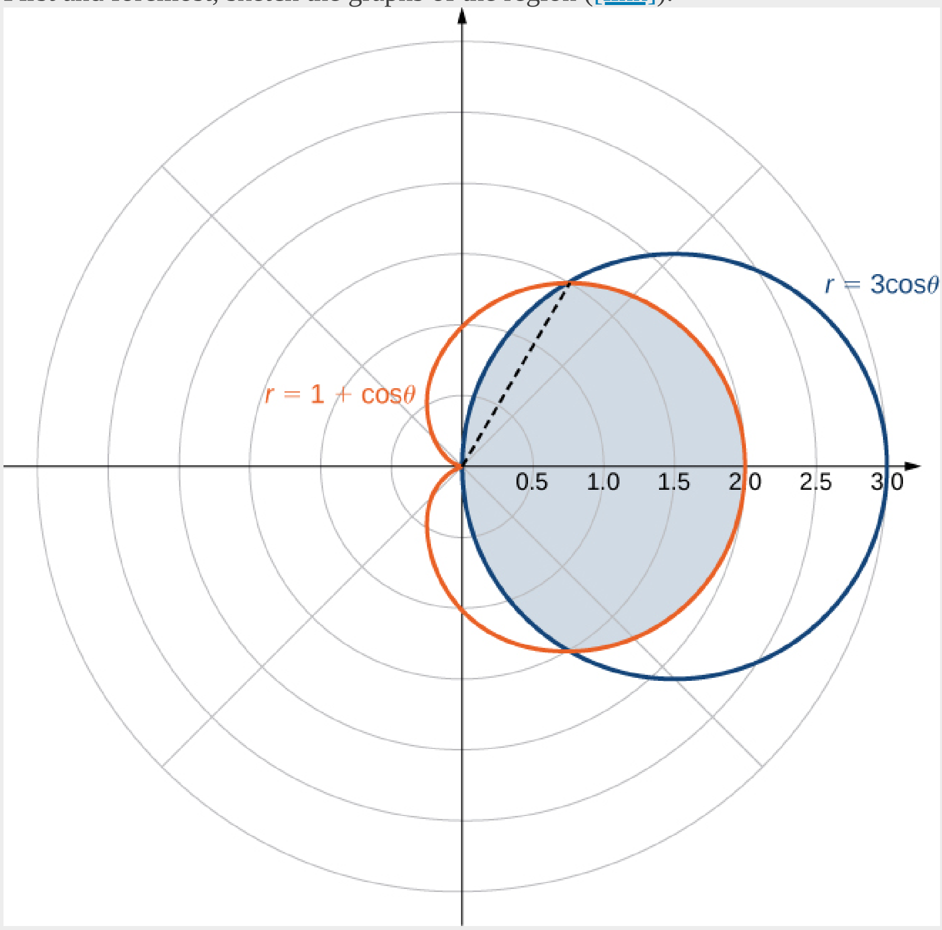
Problem:

Finding Area Between Two Polar Curves

Find the area enclosed by the circle $r = 3\cos\theta$ and the cardioid $r = 1 + \cos\theta$.

Solution:

First and foremost, sketch the graphs of the region ([link]).



Finding the area enclosed by both a circle and a cardioid.

We can from see the symmetry of the graph that we need to find the points of intersection. Setting the two equations equal to each other gives

Equation:

$$3\cos\theta = 1 + \cos\theta.$$

One of the points of intersection is $\theta = \pi/3$. The area above the polar axis consists of two parts, with one part defined by the cardioid from $\theta = 0$ to $\theta = \pi/3$ and the other part defined by the circle from $\theta = \pi/3$ to $\theta = \pi/2$. By symmetry, the total area is twice the area above the polar axis. Thus, we have

Equation:

$$A = 2\left[\int_{\theta=0}^{\theta=\pi/3}\int_{r=0}^{r=1+\cos\theta} 1r\,dr\,d\theta + \int_{\theta=\pi/3}^{\theta=\pi/2}\int_{r=0}^{r=3\cos\theta} 1r\,dr\,d\theta\right].$$

Evaluating each piece separately, we find that the area is

Equation:

$$A = 2\left(\frac{1}{4}\pi + \frac{9}{16}\sqrt{3} + \frac{3}{8}\pi - \frac{9}{16}\sqrt{3}\right) = 2\left(\frac{5}{8}\pi\right) = \frac{5}{4}\pi \text{ square units.}$$

Note:

Exercise:

Problem:

Find the area enclosed inside the cardioid $r = 3 - 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.

Solution:

$$A = 2\int_{-\pi/2}^{\pi/6}\int_{1+\sin\theta}^{3-3\sin\theta} r\,dr\,d\theta = 8\pi + 9\sqrt{3}$$

Hint

Sketch the graph, and solve for the points of intersection.

Example:

Exercise:

Problem:

Evaluating an Improper Double Integral in Polar Coordinates

Evaluate the integral $\iint\limits_{\mathrm{R}^2} e^{-10\left(x^2+y^2\right)}dx\,dy.$

Solution:

This is an improper integral because we are integrating over an unbounded region R^2. In polar coordinates, the entire plane R^2 can be seen as $0 \le \theta \le 2\pi$, $0 \le r \le \infty$.

Using the changes of variables from rectangular coordinates to polar coordinates, we have

Equation:

$$\begin{aligned}
\iint\limits_{R^2} e^{-10(x^2+y^2)}dx\,dy &= \int\limits_{\theta=0}^{\theta=2\pi}\int\limits_{r=0}^{r=\infty} e^{-10r^2}r\,dr\,d\theta = \int\limits_{\theta=0}^{\theta=2\pi}\left(\lim_{a\to\infty}\int\limits_{r=0}^{r=a} e^{-10r^2}r\,dr\right)d\theta \\
&= \left(\int\limits_{\theta=0}^{\theta=2\pi} d\theta\right)\left(\lim_{a\to\infty}\int\limits_{r=0}^{r=a} e^{-10r^2}r\,dr\right) \\
&= 2\pi\left(\lim_{a\to\infty}\int\limits_{r=0}^{r=a} e^{-10r^2}r\,dr\right) \\
&= 2\pi\lim_{a\to\infty}\left(-\tfrac{1}{20}\right)\left(e^{-10r^2}\Big|_0^a\right) \\
&= 2\pi\left(-\tfrac{1}{20}\right)\lim_{a\to\infty}\left(e^{-10a^2}-1\right) \\
&= \tfrac{\pi}{10}.
\end{aligned}$$

Note:
Exercise:

Problem: Evaluate the integral $\iint\limits_{R^2} e^{-4(x^2+y^2)}dx\,dy$.

Solution:

$\frac{\pi}{4}$

Hint

Convert to the polar coordinate system.

Key Concepts

- To apply a double integral to a situation with circular symmetry, it is often convenient to use a double integral in polar coordinates. We can apply these double integrals over a polar rectangular region or a general polar region, using an iterated integral similar to those used with rectangular double integrals.
- The area dA in polar coordinates becomes $r\,dr\,d\theta$.
- Use $x = r\cos\theta$, $y = r\sin\theta$, and $dA = r\,dr\,d\theta$ to convert an integral in rectangular coordinates to an integral in polar coordinates.
- Use $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ to convert an integral in polar coordinates to an integral in rectangular coordinates, if needed.
- To find the volume in polar coordinates bounded above by a surface $z = f(r,\theta)$ over a region on the xy-plane, use a double integral in polar coordinates.

Key Equations

- **Double integral over a polar rectangular region** R
$$\iint\limits_R f(r,\theta)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}*, \theta_{ij}*)\Delta A = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}*, \theta_{ij}*) r_{ij}* \Delta r \Delta\theta$$
- **Double integral over a general polar region**
$$\iint\limits_D f(r,\theta) r\, dr\, d\theta = \int\limits_{\theta=\alpha}^{\theta=\beta} \int\limits_{r=h_1(\theta)}^{r=h_2(\theta)} f(r,\theta) r\, dr\, d\theta$$

In the following exercises, express the region D in polar coordinates.
Exercise:

Problem: D is the region of the disk of radius 2 centered at the origin that lies in the first quadrant.

Exercise:

Problem:

D is the region between the circles of radius 4 and radius 5 centered at the origin that lies in the second quadrant.

Solution:

$D = \left\{(r,\theta) | 4 \le r \le 5, \frac{\pi}{2} \le \theta \le \pi\right\}$

Exercise:

Problem: D is the region bounded by the y-axis and $x = \sqrt{1-y^2}$.

Exercise:

Problem: D is the region bounded by the x-axis and $y = \sqrt{2-x^2}$.

Solution:

$D = \left\{(r,\theta) | 0 \le r \le \sqrt{2}, 0 \le \theta \le \pi\right\}$

Exercise:

Problem: $D = \left\{(x,y) | x^2 + y^2 \le 4x\right\}$

Exercise:

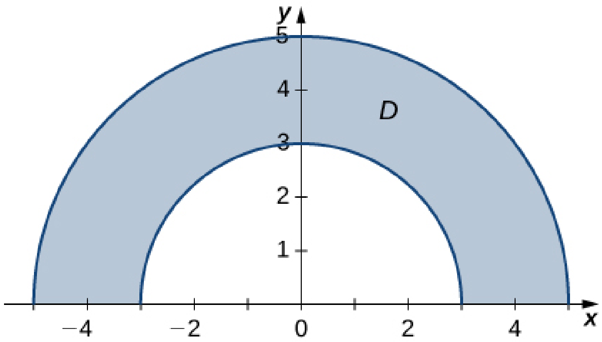
Problem: $D = \left\{(x,y) | x^2 + y^2 \le 4y\right\}$

Solution:

$D = \{(r,\theta) | 0 \le r \le 4\sin\theta, 0 \le \theta \le \pi\}$

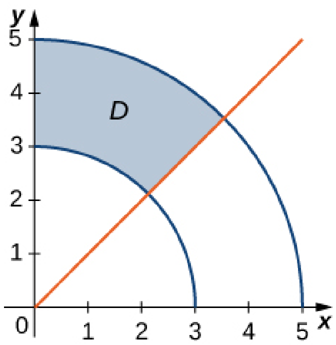
In the following exercises, the graph of the polar rectangular region D is given. Express D in polar coordinates.
Exercise:

Problem:



Exercise:

Problem:

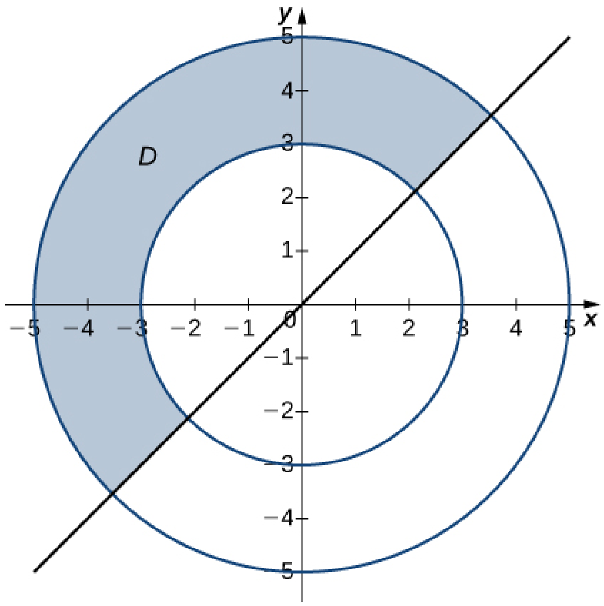


Solution:

$D = \left\{(r,\theta)|3 \le r \le 5, \frac{\pi}{4} \le \theta \le \frac{\pi}{2}\right\}$

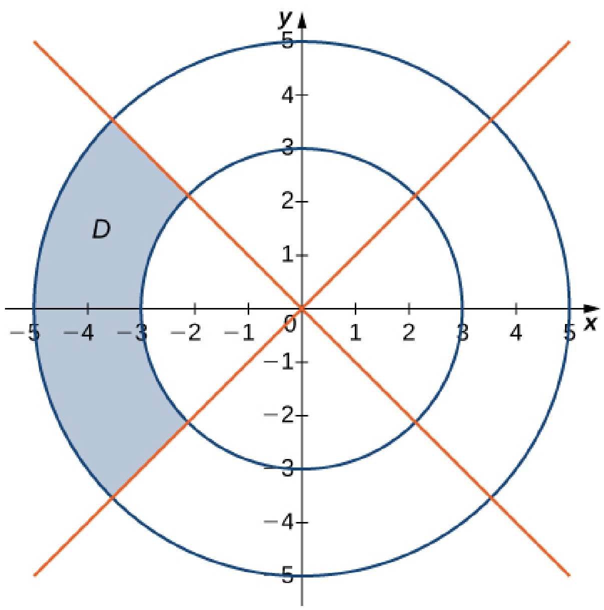
Exercise:

Problem:



Exercise:

Problem:



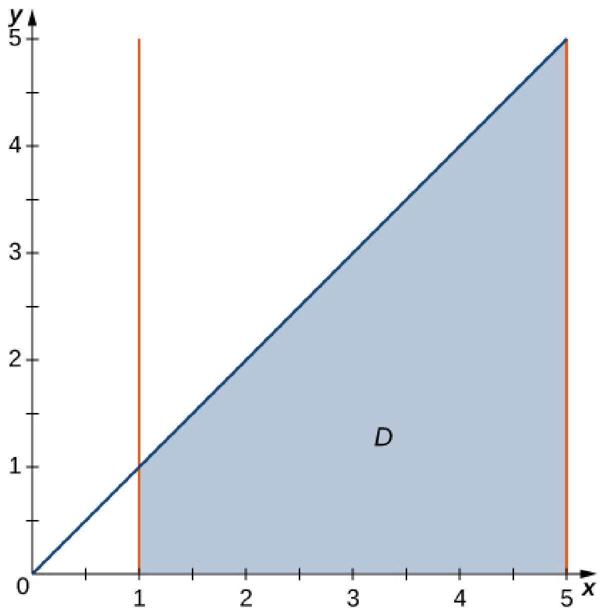
Solution:

$D = \left\{(r, \theta) | 3 \le r \le 5, \frac{3\pi}{4} \le \theta \le \frac{5\pi}{4}\right\}$

Exercise:

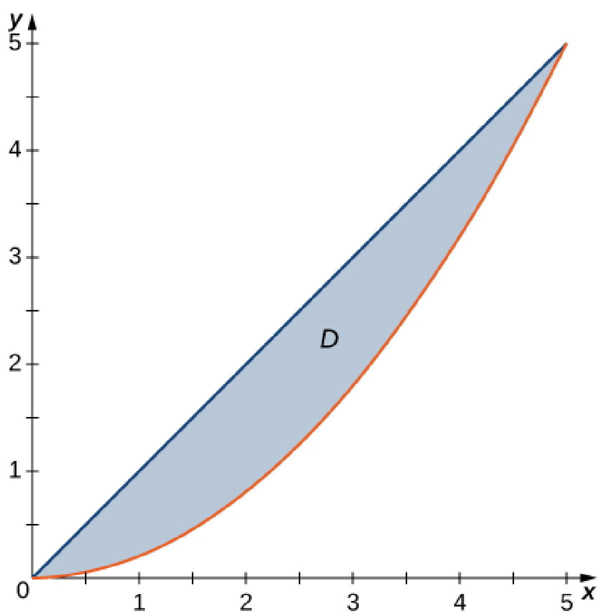
Problem:

In the following graph, the region D is situated below $y = x$ and is bounded by $x = 1, x = 5$, and $y = 0$.



Exercise:

Problem: In the following graph, the region D is bounded by $y = x$ and $y = x^2$.



Solution:

$D = \left\{ (r, \theta) | 0 \leq r \leq \tan\theta \sec\theta, 0 \leq \theta \leq \frac{\pi}{4} \right\}$

In the following exercises, evaluate the double integral $\iint\limits_R f(x, y) dA$ over the polar rectangular region D.

Exercise:

Problem: $f\left(x,y\right)=x^2+y^2, D=\left\{\left(r,\theta\right)|3\le r\le 5, 0\le \theta \le 2\pi\right\}$

Exercise:

Problem: $f\left(x,y\right)=x+y,\ \ D=\left\{\left(r,\theta\right)|3\le r\le 5, 0\le \theta \le 2\pi\right\}$

Solution:

0

Exercise:

Problem: $f\left(x,y\right)=x^2+xy, D=\left\{\left(r,\theta\right)|1\le r\le 2, \pi \le \theta \le 2\pi\right\}$

Exercise:

Problem: $f\left(x,y\right)=x^4+y^4, D=\left\{\left(r,\theta\right)|1\le r\le 2, \frac{3\pi}{2}\le \theta \le 2\pi\right\}$

Solution:

$\frac{63\pi}{16}$

Exercise:

Problem: $f\left(x,y\right)=\sqrt[3]{x^2+y^2}$, where $D=\left\{\left(r,\theta\right)|0\le r\le 1, \frac{\pi}{2}\le \theta \le \pi\right\}$.

Exercise:

Problem: $f\left(x,y\right)=x^4+2x^2y^2+y^4$, where $D=\left\{\left(r,\theta\right)|3\le r\le 4, \frac{\pi}{3}\le \theta \le \frac{2\pi}{3}\right\}$.

Solution:

$\frac{3367\pi}{18}$

Exercise:

Problem: $f(x,y)=\sin\left(\arctan\frac{y}{x}\right)$, where $D=\left\{\left(r,\theta\right)\middle|1\le r\le 2, \frac{\pi}{6}\le \theta \le \frac{\pi}{3}\right\}$

Exercise:

Problem: $f(x,y)=\arctan\left(\frac{y}{x}\right)$, where $D=\left\{\left(r,\theta\right)\middle|2\le r\le 3, \frac{\pi}{4}\le \theta \le \frac{\pi}{3}\right\}$

Solution:

$\frac{35\pi^2}{576}$

Exercise:

Problem: $\underset{D}{\iint} e^{x^2+y^2}\left[1+2\arctan\left(\frac{y}{x}\right)\right]dA, D=\left\{\left(r,\theta\right)|1\le r\le 2, \frac{\pi}{6}\le \theta \le \frac{\pi}{3}\right\}$

Exercise:

Problem: $\iint\limits_D \left(e^{x^2+y^2} + x^4 + 2x^2y^2 + y^4\right) \arctan\left(\frac{y}{x}\right) dA,\ D = \left\{(r,\theta) | 1 \le r \le 2, \frac{\pi}{4} \le \theta \le \frac{\pi}{3}\right\}$

Solution:

$\frac{7}{576}\pi^2\left(21 - e + e^4\right)$

In the following exercises, the integrals have been converted to polar coordinates. Verify that the identities are true and choose the easiest way to evaluate the integrals, in rectangular or polar coordinates.
Exercise:

Problem: $\int\limits_1^2 \int\limits_0^x \left(x^2+y^2\right) dy\, dx = \int\limits_0^{\frac{\pi}{4}} \int\limits_{\sec\theta}^{2\sec\theta} r^3 dr\, d\theta$

Exercise:

Problem: $\int\limits_2^3 \int\limits_0^x \frac{x}{\sqrt{x^2+y^2}} dy\, dx = \int\limits_0^{\pi/4} \int\limits_0^{\tan\theta\sec\theta} r\cos\theta\, dr\, d\theta$

Solution:

$\frac{5}{4}\ln\left(3 + 2\sqrt{2}\right)$

Exercise:

Problem: $\int\limits_0^1 \int\limits_{x^2}^x \frac{1}{\sqrt{x^2+y^2}} dy\, dx = \int\limits_0^{\pi/4} \int\limits_0^{\tan\theta\sec\theta} dr\, d\theta$

Exercise:

Problem: $\int\limits_0^1 \int\limits_{x^2}^x \frac{y}{\sqrt{x^2+y^2}} dy\, dx = \int\limits_0^{\pi/4} \int\limits_0^{\tan\theta\sec\theta} r\sin\theta\, dr\, d\theta$

Solution:

$\frac{1}{6}\left(2 - \sqrt{2}\right)$

In the following exercises, convert the integrals to polar coordinates and evaluate them.
Exercise:

Problem: $$\int_0^3 \int_0^{\sqrt{9-y^2}} \left(x^2+y^2\right) dx\, dy$$

Exercise:

Problem: $$\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \left(x^2+y^2\right)^2 dx\, dy$$

Solution:

$$\int_0^{\pi} \int_0^{2} r^5 dr\, d\theta = \frac{32\pi}{3}$$

Exercise:

Problem: $$\int_0^1 \int_0^{\sqrt{1-x^2}} (x+y) dy\, dx$$

Exercise:

Problem: $$\int_0^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \sin\left(x^2+y^2\right) dy\, dx$$

Solution:

$$\int_{-\pi/2}^{\pi/2} \int_0^{4} r \sin\left(r^2\right) dr\, d\theta = \pi \sin^2 8$$

Exercise:

Problem:

Evaluate the integral $\iint_D r\, dA$ where D is the region bounded by the polar axis and the upper half of the cardioid $r = 1 + \cos\theta$.

Exercise:

Problem:

Find the area of the region D bounded by the polar axis and the upper half of the cardioid $r = 1 + \cos\theta$.

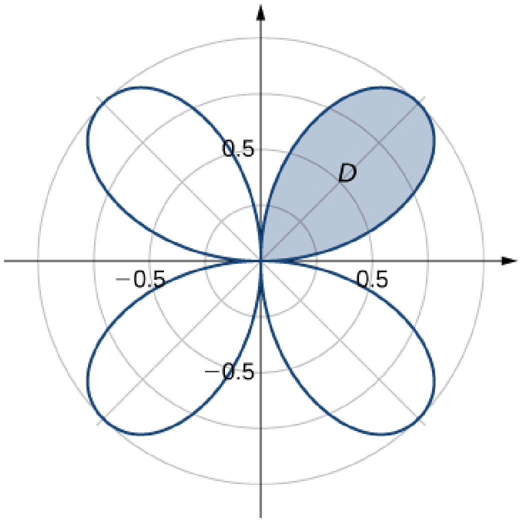
Solution:

$\frac{3\pi}{4}$

Exercise:

Problem:

Evaluate the integral $\iint_D r\,dA$, where D is the region bounded by the part of the four-leaved rose $r = \sin 2\theta$ situated in the first quadrant (see the following figure).



Exercise:

Problem:

Find the total area of the region enclosed by the four-leaved rose $r = \sin 2\theta$ (see the figure in the previous exercise).

Solution:

$\frac{\pi}{2}$

Exercise:

Problem:

Find the area of the region D, which is the region bounded by $y = \sqrt{4 - x^2}$, $x = \sqrt{3}$, $x = 2$, and $y = 0$.

Exercise:

Problem:

Find the area of the region D, which is the region inside the disk $x^2 + y^2 \leq 4$ and to the right of the line $x = 1$.

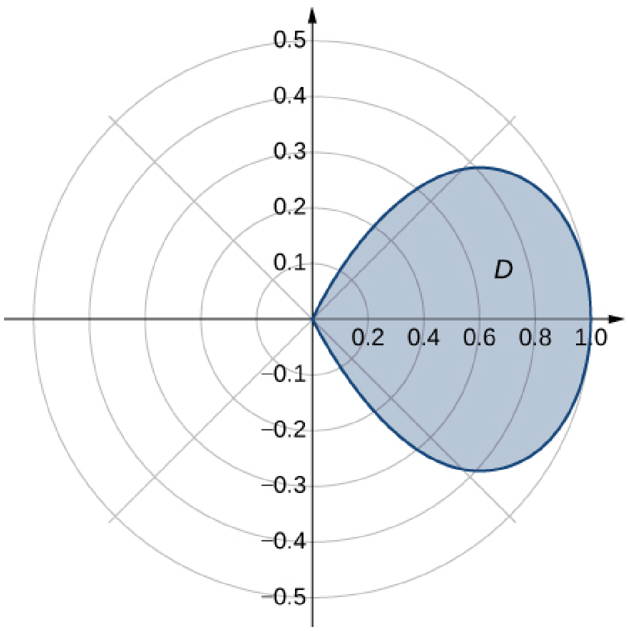
Solution:

$\frac{1}{3}\left(4\pi - 3\sqrt{3}\right)$

Exercise:

Problem:

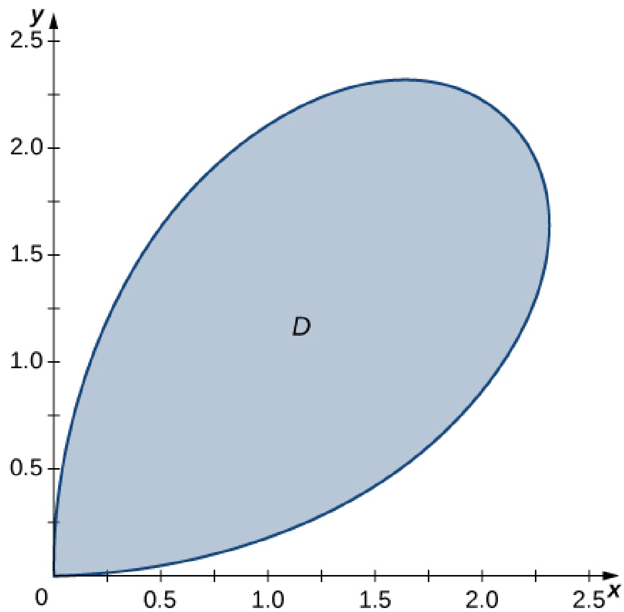
Determine the average value of the function $f\left(x,y\right)=x^2+y^2$ over the region D bounded by the polar curve $r=\cos 2\theta$, where $-\frac{\pi}{4}\le\theta\le\frac{\pi}{4}$ (see the following graph).



Exercise:

Problem:

Determine the average value of the function $f\left(x,y\right)=\sqrt{x^2+y^2}$ over the region D bounded by the polar curve $r=3\sin 2\theta$, where $0\le\theta\le\frac{\pi}{2}$ (see the following graph).



Solution:

$\frac{16}{3\pi}$

Exercise:

Problem:

Find the volume of the solid situated in the first octant and bounded by the paraboloid $z = 1 - 4x^2 - 4y^2$ and the planes $x = 0, y = 0$, and $z = 0$.

Exercise:

Problem:

Find the volume of the solid bounded by the paraboloid $z = 2 - 9x^2 - 9y^2$ and the plane $z = 1$.

Solution:

$\frac{\pi}{18}$

Exercise:

Problem:

a. Find the volume of the solid S_1 bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = 0$ and $z = 1$.
b. Find the volume of the solid S_2 outside the double cone $z^2 = x^2 + y^2$, inside the cylinder $x^2 + y^2 = 1$, and above the plane $z = 0$.
c. Find the volume of the solid inside the cone $z^2 = x^2 + y^2$ and below the plane $z = 1$ by subtracting the volumes of the solids S_1 and S_2.

Exercise:

Problem:

a. Find the volume of the solid S_1 inside the unit sphere $x^2 + y^2 + z^2 = 1$ and above the plane $z = 0$.
b. Find the volume of the solid S_2 inside the double cone $(z-1)^2 = x^2 + y^2$ and above the plane $z = 0$.
c. Find the volume of the solid outside the double cone $(z-1)^2 = x^2 + y^2$ and inside the sphere $x^2 + y^2 + z^2 = 1$.

Solution:

a. $\frac{2\pi}{3}$; b. $\frac{\pi}{2}$; c. $\frac{\pi}{6}$

For the following two exercises, consider a spherical ring, which is a sphere with a cylindrical hole cut so that the axis of the cylinder passes through the center of the sphere (see the following figure).

Exercise:

Problem:

If the sphere has radius 4 and the cylinder has radius 2, find the volume of the spherical ring.

Exercise:

Problem:

A cylindrical hole of diameter 6 cm is bored through a sphere of radius 5 cm such that the axis of the cylinder passes through the center of the sphere. Find the volume of the resulting spherical ring.

Solution:

$\frac{256\pi}{3}\ \text{cm}^3$

Exercise:

Problem:

Find the volume of the solid that lies under the double cone $z^2 = 4x^2 + 4y^2$, inside the cylinder $x^2 + y^2 = x$, and above the plane $z = 0$.

Exercise:

Problem:

Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, inside the cylinder $x^2 + y^2 = x$, and above the plane $z = 0$.

Solution:

$\frac{3\pi}{32}$

Exercise:

Problem:

Find the volume of the solid that lies under the plane $x + y + z = 10$ and above the disk $x^2 + y^2 = 4x$.

Exercise:

Problem:

Find the volume of the solid that lies under the plane $2x + y + 2z = 8$ and above the unit disk $x^2 + y^2 = 1$.

Solution:

4π

Exercise:

Problem:

A radial function f is a function whose value at each point depends only on the distance between that point and the origin of the system of coordinates; that is, $f(x, y) = g(r)$, where $r = \sqrt{x^2 + y^2}$. Show that if f is a continuous radial function, then $\iint\limits_D f(x, y) dA = (\theta_2 - \theta_1)\left[G(R_2) - G(R_1)\right]$, where $G\prime(r) = rg(r)$ and $(x, y) \in D = \{(r, \theta) | R_1 \le r \le R_2, 0 \le \theta \le 2\pi\}$, with $0 \le R_1 < R_2$ and $0 \le \theta_1 < \theta_2 \le 2\pi$.

Exercise:

Problem:

Use the information from the preceding exercise to calculate the integral $\iint\limits_D \left(x^2 + y^2\right)^3 dA$, where D is the unit disk.

Solution:

$\frac{\pi}{4}$

Exercise:

Problem:

Let $f(x, y) = \frac{F\prime(r)}{r}$ be a continuous radial function defined on the annular region $D = \{(r, \theta) | R_1 \leq r \leq R_2, 0 \leq \theta \leq 2\pi\}$, where $r = \sqrt{x^2 + y^2}$, $0 < R_1 < R_2$, and F is a differentiable function. Show that $\iint\limits_D f(x, y) dA = 2\pi [F(R_2) - F(R_1)]$.

Exercise:

Problem:

Apply the preceding exercise to calculate the integral $\iint\limits_D \frac{e^{\sqrt{x^2+y^2}}}{\sqrt{x^2 + y^2}} dx\, dy$, where D is the annular region between the circles of radii 1 and 2 situated in the third quadrant.

Solution:

$\frac{1}{2} \pi e(e - 1)$

Exercise:

Problem:

Let f be a continuous function that can be expressed in polar coordinates as a function of θ only; that is, $f(x, y) = h(\theta)$, where $(x, y) \in D = \{(r, \theta) | R_1 \leq r \leq R_2, \theta_1 \leq \theta \leq \theta_2\}$, with $0 \leq R_1 < R_2$ and $0 \leq \theta_1 < \theta_2 \leq 2\pi$. Show that $\iint\limits_D f(x, y) dA = \frac{1}{2} \left(R_2^2 - R_1^2\right) [H(\theta_2) - H(\theta_1)]$, where H is an antiderivative of h.

Exercise:

Problem:

Apply the preceding exercise to calculate the integral $\iint\limits_D \frac{y^2}{x^2} dA$, where $D = \left\{(r, \theta) \middle| 1 \leq r \leq 2, \frac{\pi}{6} \leq \theta \leq \frac{\pi}{3}\right\}$.

Solution:

$\sqrt{3} - \frac{\pi}{4}$

Exercise:

Problem:

Let f be a continuous function that can be expressed in polar coordinates as a function of θ only; that is, $f(x, y) = g(r)h(\theta)$, where $(x, y) \in D = \{(r, \theta) | R_1 \leq r \leq R_2, \theta_1 \leq \theta \leq \theta_2\}$ with $0 \leq R_1 < R_2$ and $0 \leq \theta_1 < \theta_2 \leq 2\pi$. Show that $\iint\limits_D f(x, y) dA = [G(R_2) - G(R_1)]\ [H(\theta_2) - H(\theta_1)]$, where G and H are antiderivatives of g and h, respectively.

Exercise:

Problem: Evaluate $\iint\limits_D \arctan\left(\frac{y}{x}\right)\sqrt{x^2+y^2}dA$, where $D = \left\{(r,\theta) \middle| 2 \le r \le 3, \frac{\pi}{4} \le \theta \le \frac{\pi}{3}\right\}$.

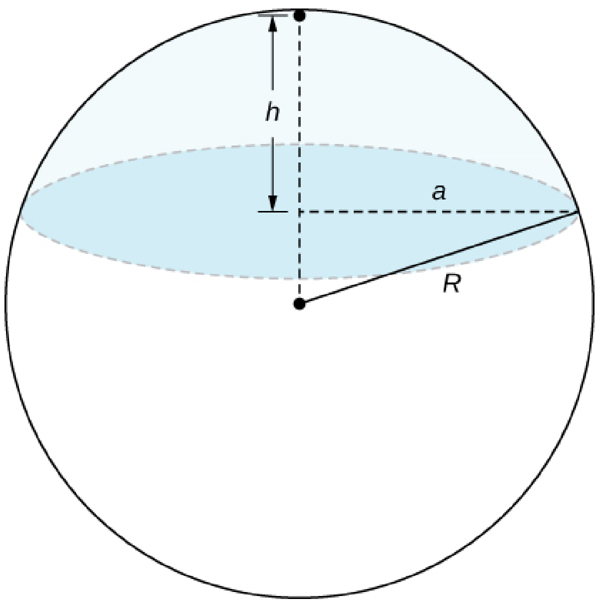
Solution:

$\frac{133\pi^3}{864}$

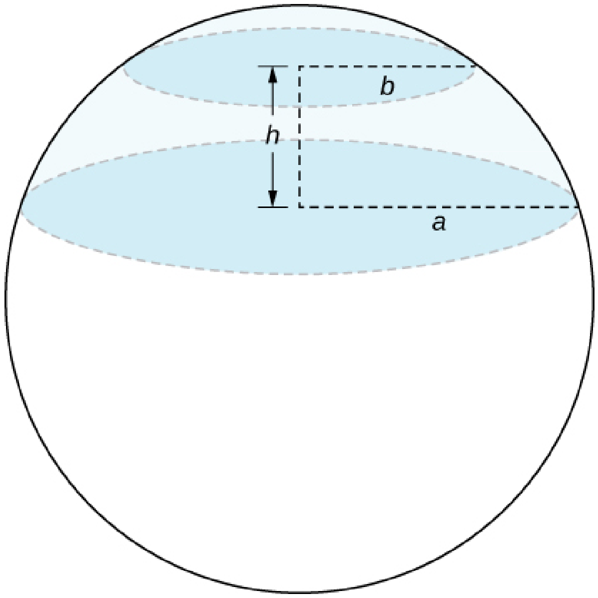
Exercise:

Problem: A spherical cap is the region of a sphere that lies above or below a given plane.

a. Show that the volume of the spherical cap in the figure below is $\frac{1}{6}\pi h\left(3a^2+h^2\right)$.



b. A spherical segment is the solid defined by intersecting a sphere with two parallel planes. If the distance between the planes is h, show that the volume of the spherical segment in the figure below is $\frac{1}{6}\pi h\left(3a^2+3b^2+h^2\right)$.



Exercise:

Problem:

In statistics, the joint density for two independent, normally distributed events with a mean $\mu = 0$ and a standard distribution σ is defined by $p(x, y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$. Consider (X, Y), the Cartesian coordinates of a ball in the resting position after it was released from a position on the *z*-axis toward the xy-plane. Assume that the coordinates of the ball are independently normally distributed with a mean $\mu = 0$ and a standard deviation of σ (in feet). The probability that the ball will stop no more than a feet from the origin is given by $P\left[X^2 + Y^2 \leq a^2\right] = \iint\limits_D p(x, y) dy\, dx$, where D is the disk of radius *a* centered at the origin. Show that $P\left[X^2 + Y^2 \leq a^2\right] = 1 - e^{-a^2/2\sigma^2}$.

Exercise:

Problem:

The double improper integral $\int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} e^{(-x^2+y^2/2)} dy\, dx$ may be defined as the limit value of the double integrals $\iint\limits_{D_a} e^{(-x^2+y^2/2)} dA$ over disks D_a of radii *a* centered at the origin, as *a* increases without bound; that is, $\int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} e^{(-x^2+y^2/2)} dy\, dx = \lim\limits_{a \to \infty} \iint\limits_{D_a} e^{(-x^2+y^2/2)} dA$.

a. Use polar coordinates to show that $\int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} e^{(-x^2+y^2/2)} dy\, dx = 2\pi$.

b. Show that $\int\limits_{-\infty}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi}$, by using the relation

$$\int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} e^{(-x^2+y^2/2)} dy\, dx = \left(\int\limits_{-\infty}^{\infty} e^{-x^2/2} dx \right) \left(\int\limits_{-\infty}^{\infty} e^{-y^2/2} dy \right).$$

Glossary

polar rectangle

the region enclosed between the circles $r = a$ and $r = b$ and the angles $\theta = \alpha$ and $\theta = \beta$; it is described as $R = \{(r, \theta) | a \le r \le b, \alpha \le \theta \le \beta\}$

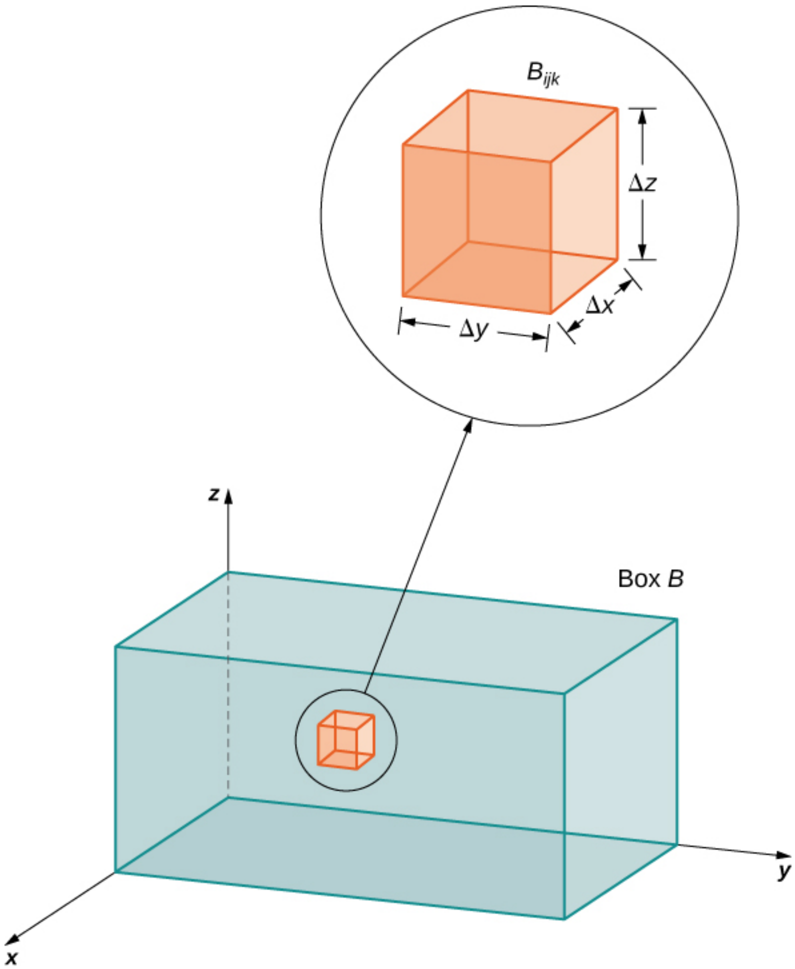
Triple Integrals

- Recognize when a function of three variables is integrable over a rectangular box.
- Evaluate a triple integral by expressing it as an iterated integral.
- Recognize when a function of three variables is integrable over a closed and bounded region.
- Simplify a calculation by changing the order of integration of a triple integral.
- Calculate the average value of a function of three variables.

In Double Integrals over Rectangular Regions, we discussed the double integral of a function $f(x, y)$ of two variables over a rectangular region in the plane. In this section we define the triple integral of a function $f(x, y, z)$ of three variables over a rectangular solid box in space, $\mathbb{R}^3$. Later in this section we extend the definition to more general regions in $\mathbb{R}^3$.

Integrable Functions of Three Variables

We can define a rectangular box B in $\mathbb{R}^3$ as $B = \{(x, y, z) | a \leq x \leq b, c \leq y \leq d, e \leq z \leq f\}$. We follow a similar procedure to what we did in Double Integrals over Rectangular Regions. We divide the interval $[a, b]$ into l subintervals $[x_{i-1}, x_i]$ of equal length $\Delta x = \frac{x_i - x_{i-1}}{l}$, divide the interval $[c, d]$ into m subintervals $[y_{i-1}, y_i]$ of equal length $\Delta y = \frac{y_j - y_{j-1}}{m}$, and divide the interval $[e, f]$ into n subintervals $[z_{i-1}, z_i]$ of equal length $\Delta z = \frac{z_k - z_{k-1}}{n}$. Then the rectangular box B is subdivided into lmn subboxes $B_{ijk} = [x_{i-1}, x_i] \times [y_{i-1}, y_i] \times [z_{i-1}, z_i]$, as shown in [link].



A rectangular box in $\mathbb{R}^3$ divided into subboxes by planes parallel to the coordinate planes.

For each $i, j,$ and k, consider a sample point $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ in each sub-box B_{ijk}. We see that its volume is $\Delta V = \Delta x \Delta y \Delta z$. Form the triple Riemann sum

Equation:

$$\sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \Delta x \Delta y \Delta z.$$

We define the triple integral in terms of the limit of a triple Riemann sum, as we did for the double integral in terms of a double Riemann sum.

Note:

Definition

The **triple integral** of a function $f(x, y, z)$ over a rectangular box B is defined as

Equation:

$$\lim_{l,m,n\to\infty}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}^{*}, y_{ijk}^{*}, z_{ijk}^{*})\Delta x\Delta y\Delta z = \iiint_B f(x,y,z)dV$$

if this limit exists.

When the triple integral exists on B, the function $f(x,y,z)$ is said to be integrable on B. Also, the triple integral exists if $f(x,y,z)$ is continuous on B. Therefore, we will use continuous functions for our examples. However, continuity is sufficient but not necessary; in other words, f is bounded on B and continuous except possibly on the boundary of B. The sample point $(x_{ijk}^{*}, y_{ijk}^{*}, z_{ijk}^{*})$ can be any point in the rectangular sub-box B_{ijk} and all the properties of a double integral apply to a triple integral. Just as the double integral has many practical applications, the triple integral also has many applications, which we discuss in later sections.

Now that we have developed the concept of the triple integral, we need to know how to compute it. Just as in the case of the double integral, we can have an iterated triple integral, and consequently, a version of Fubini's thereom for triple integrals exists.

Note:
Fubini's Theorem for Triple Integrals
If $f(x,y,z)$ is continuous on a rectangular box $B = [a,b] \times [c,d] \times [e,f]$, then
Equation:

$$\iiint_B f(x,y,z)dV = \int_e^f \int_c^d \int_a^b f(x,y,z)dx\,dy\,dz.$$

This integral is also equal to any of the other five possible orderings for the iterated triple integral.

For a, b, c, d, e, and f real numbers, the iterated triple integral can be expressed in six different orderings:
Equation:

$$\begin{aligned}\int_e^f \int_c^d \int_a^b f(x,y,z)dx\,dy\,dz &= \int_e^f \left(\int_c^d \left(\int_a^b f(x,y,z)dx\right)dy\right)dz = \int_c^d \left(\int_e^f \left(\int_a^b f(x,y,z)dx\right)dz\right)dy \\ &= \int_a^b \left(\int_e^f \left(\int_c^d f(x,y,z)dy\right)dz\right)dx = \int_e^f \left(\int_a^b \left(\int_c^d f(x,y,z)dy\right)dx\right)dz \\ &= \int_c^e \left(\int_a^b \left(\int_e^f f(x,y,z)dz\right)dx\right)dy = \int_a^b \left(\int_c^e \left(\int_e^f f(x,y,z)dz\right)dy\right)dx.\end{aligned}$$

For a rectangular box, the order of integration does not make any significant difference in the level of difficulty in computation. We compute triple integrals using Fubini's Theorem rather than using the Riemann sum definition. We follow the order of integration in the same way as we did for double integrals (that is, from inside to outside).

Example:

Exercise:

Problem:
Evaluating a Triple Integral

Evaluate the triple integral $\int_{z=0}^{z=1} \int_{y=2}^{y=4} \int_{x=-1}^{x=5} (x + yz^2) dx\, dy\, dz.$

Solution:

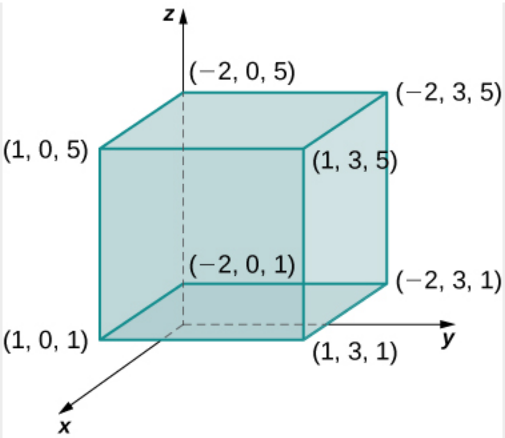
The order of integration is specified in the problem, so integrate with respect to x first, then y, and then z.
Equation:

$$\begin{aligned} & \int_{z=0}^{z=1} \int_{y=2}^{y=4} \int_{x=-1}^{x=5} (x + yz^2) dx\, dy\, dz & \\ = & \int_{z=0}^{z=1} \int_{y=2}^{y=4} \left[\frac{x^2}{2} + xyz^2 \Big|_{x=-1}^{x=5} \right] dy\, dz & \text{Integrate with respect to } x. \\ = & \int_{z=0}^{z=1} \int_{y=2}^{y=4} \left[12 + 6yz^2 \right] dy\, dz & \text{Evaluate.} \\ = & \int_{z=0}^{z=1} \left[12y + 6\frac{y^2}{2} z^2 \Big|_{y=2}^{y=4} \right] dz & \text{Integrate with respect to } y. \\ = & \int_{z=0}^{z=1} \left[24 + 36z^2 \right] dz & \text{Evaluate.} \\ = & \left[24z + 36\tfrac{z^3}{3} \right]_{z=0}^{z=1} = 36. & \text{Integrate with respect to } z. \end{aligned}$$

Example:
Exercise:

Problem:
Evaluating a Triple Integral

Evaluate the triple integral $\iiint\limits_B x^2 yz\, dV$ where $B = \{(x, y, z) | -2 \le x \le 1, 0 \le y \le 3, 1 \le z \le 5\}$ as shown in the following figure.



Evaluating a triple integral over a given rectangular box.

Solution:

The order is not specified, but we can use the iterated integral in any order without changing the level of difficulty. Choose, say, to integrate *y* first, then *x*, and then *z*.

$$\begin{aligned}\iiint_B x^2yz\,dV &= \int_1^5\int_{-2}^1\int_0^3 \left[x^2yz\right]dy\,dx\,dz = \int_1^5\int_{-2}^1 \left[x^2\tfrac{y^2}{2}z\Big|_0^3\right]dx\,dz \\ &= \int_1^5\int_{-2}^1 \tfrac{9}{2}x^2z\,dx\,dz = \int_1^5\left[\tfrac{9}{2}\tfrac{x^3}{3}z\Big|_{-2}^1\right]dz = \int_1^5 \tfrac{27}{2}z\,dz = \tfrac{27}{2}\tfrac{z^2}{2}\Big|_1^5 = 162.\end{aligned}$$

Now try to integrate in a different order just to see that we get the same answer. Choose to integrate with respect to x first, then z, and then y.
Equation:

$$\begin{aligned}\iiint_B x^2yz\,dV &= \int_0^3\int_1^5\int_{-2}^1 \left[x^2yz\right]dx\,dz\,dy = \int_0^3\int_1^5\left[\frac{x^3}{3}yz\Big|_{-2}^1\right]dz\,dy \\ &= \int_0^3\int_1^5 3yz\,dz\,dy = \int_0^3\left[3y\frac{z^2}{2}\Big|_1^5\right]dy = \int_0^3 36y\,dy = 36\tfrac{y^2}{2}\Big|_0^3 = 18(9-0) = 162.\end{aligned}$$

Note:
Exercise:

Problem:

Evaluate the triple integral $\iiint\limits_B z\sin x\cos y\,dV$ where

$B = \left\{(x,y,z)\middle|0 \le x \le \pi, \frac{3\pi}{2} \le y \le 2\pi, 1 \le z \le 3\right\}.$

Solution:

$\iiint\limits_B z\sin x\cos y\,dV = 8$

Hint

Follow the steps in the previous example.

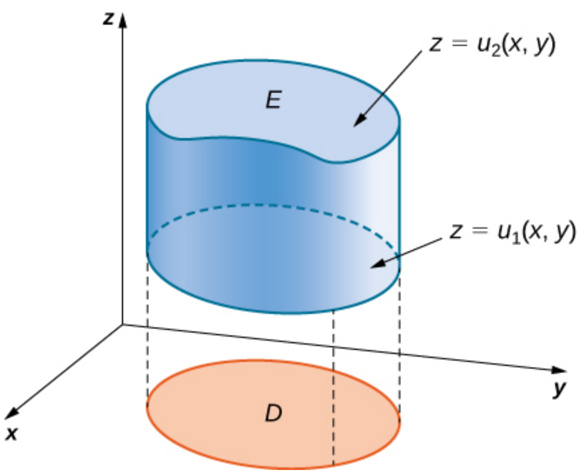
Triple Integrals over a General Bounded Region

We now expand the definition of the triple integral to compute a triple integral over a more general bounded region E in $\mathbb{R}^3$. The general bounded regions we will consider are of three types. First, let D be the bounded region that is a projection of E onto the xy-plane. Suppose the region E in $\mathbb{R}^3$ has the form

Equation:

$$E = \{(x,y,z)|(x,y) \in D, u_1(x,y) \le z \le u_2(x,y)\}.$$

For two functions $z = u_1(x,y)$ and $z = u_2(x,y)$, such that $u_1(x,y) \le u_2(x,y)$ for all (x,y) in D as shown in the following figure.



We can describe region E as the space between $u_1(x,y)$ and $u_2(x,y)$ above the projection D of E onto the xy-plane.

Note:
Triple Integral over a General Region
The triple integral of a continuous function $f(x, y, z)$ over a general three-dimensional region
Equation:

$$E = \{(x, y, z)|(x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$$

in $\mathbb{R}^3$, where D is the projection of E onto the xy-plane, is
Equation:

$$\iiint\limits_E f(x, y, z)dV = \iint\limits_D \left[\int\limits_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)dz \right] dA.$$

Similarly, we can consider a general bounded region D in the xy-plane and two functions $y = u_1(x, z)$ and $y = u_2(x, z)$ such that $u_1(x, z) \leq u_2(x, z)$ for all (x, z) in D. Then we can describe the solid region E in $\mathbb{R}^3$ as
Equation:

$$E = \{(x, y, z)|(x, z) \in D, u_1(x, z) \leq y \leq u_2(x, z)\}$$

where D is the projection of E onto the xy-plane and the triple integral is
Equation:

$$\iiint\limits_E f(x, y, z)dV = \iint\limits_D \left[\int\limits_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)dy \right] dA.$$

Finally, if D is a general bounded region in the yz-plane and we have two functions $x = u_1(y, z)$ and $x = u_2(y, z)$ such that $u_1(y, z) \leq u_2(y, z)$ for all (y, z) in D, then the solid region E in $\mathbb{R}^3$ can be described as
Equation:

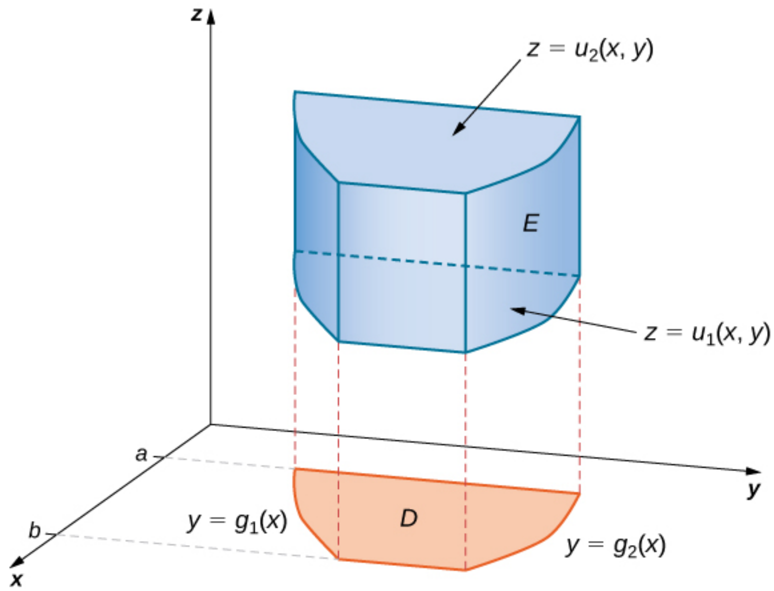
$$E = \{(x, y, z)|(y, z) \in D, u_1(y, z) \leq x \leq u_2(y, z)\}$$

where D is the projection of E onto the yz-plane and the triple integral is
Equation:

$$\iiint\limits_E f(x, y, z)dV = \iint\limits_D \left[\int\limits_{u_1(y,z)}^{u_2(y,z)} f(x, y, z)dx \right] dA.$$

Note that the region D in any of the planes may be of Type I or Type II as described in Double Integrals over General Regions. If D in the xy-plane is of Type I ([link]), then
Equation:

$$E = \{(x, y, z)|a \leq x \leq b, g_1(x) \leq y \leq g_2(x), u_1(x, y) \leq z \leq u_2(x, y)\}.$$



A box E where the projection D in the xy-plane is of Type I.

Then the triple integral becomes

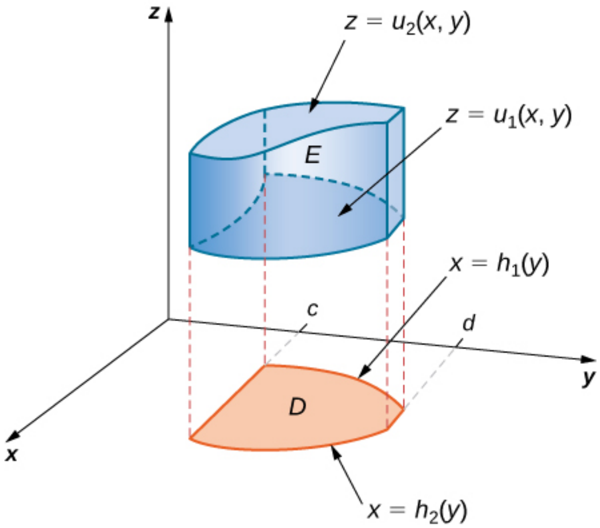
Equation:

$$\iiint_E f(x,y,z)dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dz\,dy\,dx.$$

If D in the xy-plane is of Type II ([link]), then

Equation:

$$E = \{(x,y,z)|c \le x \le d, h_1(x) \le y \le h_2(x), u_1(x,y) \le z \le u_2(x,y)\}.$$



A box E where the projection D in the xy-plane is of Type II.

Then the triple integral becomes
Equation:

$$\iiint_E f(x,y,z)dV = \int_{y=c}^{y=d} \int_{x=h_1(y)}^{x=h_2(y)} \int_{z=u_1(x,y)}^{z=u_2(x,y)} f(x,y,z)dz\,dx\,dy.$$

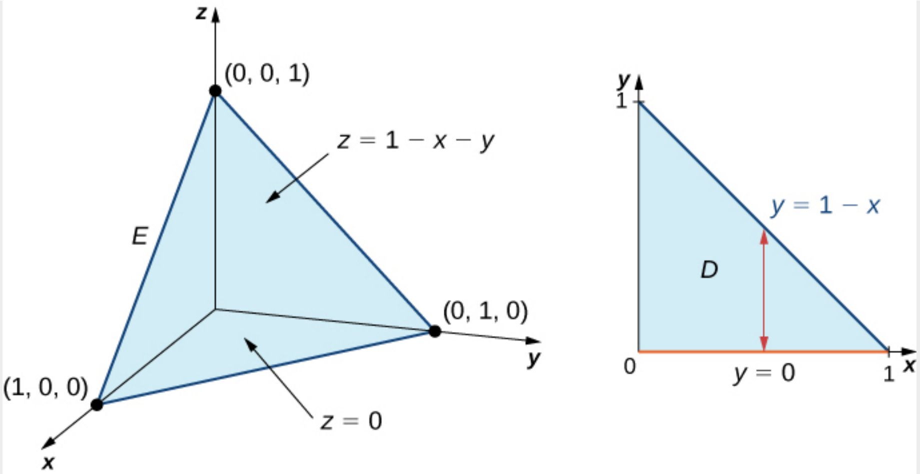
Example:
Exercise:

Problem:
Evaluating a Triple Integral over a General Bounded Region

Evaluate the triple integral of the function $f(x,y,z) = 5x - 3y$ over the solid tetrahedron bounded by the planes $x = 0, y = 0, z = 0$, and $x + y + z = 1$.

Solution:

[link] shows the solid tetrahedron E and its projection D on the xy-plane.



The solid E has a projection D on the xy-plane of Type I.

We can describe the solid region tetrahedron as
Equation:

$$E=\{(x,y,z)|0\le x\le 1, 0\le y\le 1-x, 0\le z\le 1-x-y\}.$$

Hence, the triple integral is
Equation:

$$\iiint_E f(x,y,z)dV=\int_{x=0}^{x=1}\int_{y=0}^{y=1-x}\int_{z=0}^{z=1-x-y}(5x-3y)dz\,dy\,dx.$$

To simplify the calculation, first evaluate the integral $\int_{z=0}^{z=1-x-y}(5x-3y)dz$. We have
Equation:

$$\int_{z=0}^{z=1-x-y}(5x-3y)dz=(5x-3y)(1-x-y).$$

Now evaluate the integral $\int_{y=0}^{y=1-x}(5x-3y)(1-x-y)dy$, obtaining
Equation:

$$\int_{y=0}^{y=1-x}(5x-3y)(1-x-y)dy=\tfrac{1}{2}(x-1)^2(6x-1).$$

Finally, evaluate
Equation:

$$\int_{x=0}^{x=1}\tfrac{1}{2}(x-1)^2(6x-1)dx=\tfrac{1}{12}.$$

Putting it all together, we have
Equation:

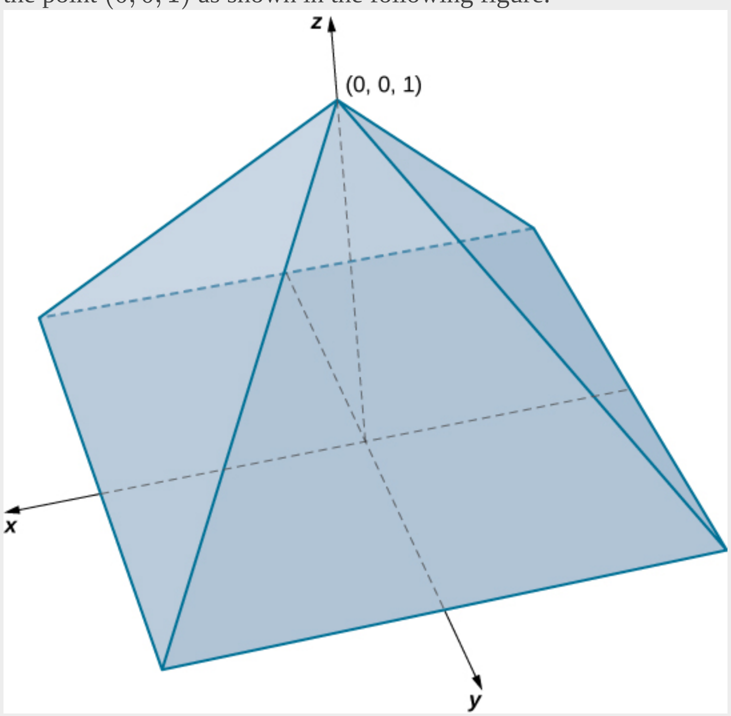
$$\iiint\limits_E f(x,y,z)dV = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x-y} (5x-3y)dz\,dy\,dx = \tfrac{1}{12}.$$

Just as we used the double integral $\iint\limits_D 1dA$ to find the area of a general bounded region D, we can use $\iiint\limits_E 1dV$ to find the volume of a general solid bounded region E. The next example illustrates the method.

Example:
Exercise:

Problem:
Finding a Volume by Evaluating a Triple Integral

Find the volume of a right pyramid that has the square base in the xy-plane $[-1,1] \times [-1,1]$ and vertex at the point $(0,0,1)$ as shown in the following figure.



Finding the volume of a pyramid with a square base.

Solution:

In this pyramid the value of z changes from 0 to 1, and at each height z, the cross section of the pyramid for any value of z is the square $[-1+z, 1-z] \times [-1+z, 1-z]$. Hence, the volume of the pyramid is $\iiint\limits_E 1dV$ where

Equation:

$$E = \{(x,y,z)|0 \le z \le 1, -1+z \le y \le 1-z, -1+z \le x \le 1-z\}.$$

Thus, we have

Equation:

$$\iiint\limits_E 1dV = \int_{z=0}^{z=1} \int_{y=1+z}^{y=1-z} \int_{x=1+z}^{x=1-z} 1dx\,dy\,dz = \int_{z=0}^{z=1} \int_{y=1+z}^{y=1-z} (2-2z)dy\,dz = \int_{z=0}^{z=1} (2-2z)^2 dz = \frac{4}{3}.$$

Hence, the volume of the pyramid is $\frac{4}{3}$ cubic units.

Note:
Exercise:

Problem:

Consider the solid sphere $E = \left\{(x,y,z)\middle|x^2+y^2+z^2 = 9\right\}$. Write the triple integral $\iiint\limits_E f(x,y,z)dV$ for an arbitrary function f as an iterated integral. Then evaluate this triple integral with $f(x,y,z) = 1$. Notice that this gives the volume of a sphere using a triple integral.

Solution:

$$\iiint\limits_E 1dV = 8\int_{x=-3}^{x=3} \int_{y=-\sqrt{9-x^2}}^{y=\sqrt{9-x^2}} \int_{z=-\sqrt{9-x^2-y^2}}^{z=\sqrt{9-x^2-y^2}} 1dz\,dy\,dx = 36\pi.$$

Hint

Follow the steps in the previous example. Use symmetry.

Changing the Order of Integration

As we have already seen in double integrals over general bounded regions, changing the order of the integration is done quite often to simplify the computation. With a triple integral over a rectangular box, the order of integration does not change the level of difficulty of the calculation. However, with a triple integral over a general bounded region, choosing an appropriate order of integration can simplify the computation quite a bit. Sometimes making the change to polar coordinates can also be very helpful. We demonstrate two examples here.

Example:
Exercise:

Problem:

Changing the Order of Integration

Consider the iterated integral

Equation:

$$\int_{x=0}^{x=1}\int_{y=0}^{y=x^2}\int_{z=0}^{z=y} f(x,y,z)dz\,dy\,dx.$$

The order of integration here is first with respect to *z*, then *y*, and then *x*. Express this integral by changing the order of integration to be first with respect to *x*, then *z*, and then *y*. Verify that the value of the integral is the same if we let $f(x,y,z) = xyz$.

Solution:

The best way to do this is to sketch the region E and its projections onto each of the three coordinate planes. Thus, let

Equation:

$$E = \{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le x^2, 0 \le z \le y\}.$$

and

Equation:

$$\int_{x=0}^{x=1}\int_{y=0}^{y=x^2}\int_{z=0}^{z=y^2} f(x,y,z)dz\,dy\,dx = \iiint_E f(x,y,z)dV.$$

We need to express this triple integral as

Equation:

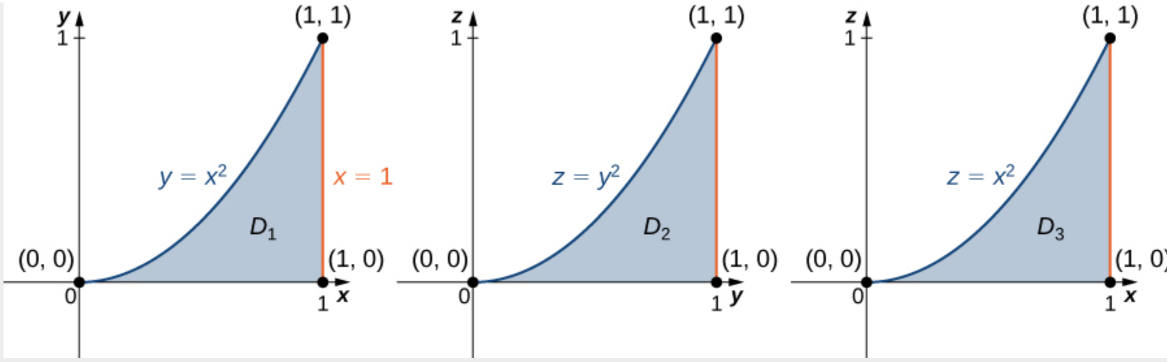
$$\int_{y=c}^{y=d}\int_{z=v_1(y)}^{z=v_2(y)}\int_{x=u_1(y,z)}^{x=u_2(y,z)} f(x,y,z)dx\,dz\,dy.$$

Knowing the region E we can draw the following projections ([link]):

on the xy-plane is $D_1 = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le x^2\} = \{(x,y) \mid 0 \le y \le 1, \sqrt{y} \le x \le 1\}$,

on the yz-plane is $D_2 = \{(y,z) \mid 0 \le y \le 1, 0 \le z \le y^2\}$, and

on the xz-plane is $D_3 = \{(x,z) \mid 0 \le x \le 1, 0 \le z \le x^2\}$.



The three cross sections of E on the three coordinate planes.

Now we can describe the same region E as $\{(x, y, z) | 0 \leq y \leq 1, 0 \leq z \leq y^2, \sqrt{y} \leq x \leq 1\}$, and consequently, the triple integral becomes

Equation:

$$\int_{y=c}^{y=d} \int_{z=v_1(y)}^{z=v_2(y)} \int_{x=u_1(y,z)}^{x=u_2(y,z)} f(x, y, z) dx\, dz\, dy = \int_{y=0}^{y=1} \int_{z=0}^{z=x^2} \int_{x=\sqrt{y}}^{x=1} f(x, y, z) dx\, dz\, dy.$$

Now assume that $f(x, y, z) = xyz$ in each of the integrals. Then we have

Equation:

$$\int_{x=0}^{x=1} \int_{y=0}^{y=x^2} \int_{z=0}^{z=y^2} xyz\, dz\, dy\, dx$$

$$= \int_{x=0}^{x=1} \int_{y=0}^{y=x^2} \left[xy \frac{z^2}{2} \Big|_{z=0}^{z=y^2} \right] dy\, dx = \int_{x=0}^{x=1} \int_{y=0}^{y=x^2} \left(x \frac{y^5}{2} \right) dy\, dx = \int_{x=0}^{x=1} \left[x \frac{y^6}{12} \Big|_{y=0}^{y=x^2} \right] dx = \int_{x=0}^{x=1} \frac{x^{13}}{12} dx = \tfrac{1}{168}$$

$$\int_{y=0}^{y=1} \int_{z=0}^{z=y^2} \int_{x=\sqrt{y}}^{x=1} xyz\, dx\, dz\, dy$$

$$= \int_{y=0}^{y=1} \int_{z=0}^{z=y^2} \left[yz \frac{x^2}{2} \Big|_{\sqrt{y}}^{1} \right] dz\, dy$$

$$= \int_{y=0}^{y=1} \int_{z=0}^{z=y^2} \left(\frac{yz}{2} - \frac{y^2 z}{2} \right) dz\, dy = \int_{y=0}^{y=1} \left[\frac{yz^2}{4} - \frac{y^2 z^2}{4} \Big|_{z=0}^{z=y^2} \right] dy = \int_{y=0}^{y=1} \left(\frac{y^5}{4} - \frac{y^6}{4} \right) dy = \tfrac{1}{168}.$$

The answers match.

Note:
Exercise:

Problem: Write five different iterated integrals equal to the given integral
Equation:

$$\int_{z=0}^{z=4}\int_{y=0}^{y=4-z}\int_{x=0}^{x=\sqrt{y}} f\left(x,y,z\right)dx\,dy\,dz.$$

Solution:

(i) $\int_{z=0}^{z=4}\int_{x=0}^{x=\sqrt{4-z}}\int_{y=x^2}^{y=4-z} f\left(x,y,z\right)dy\,dx\,dz$, (ii) $\int_{y=0}^{y=4}\int_{z=0}^{z=4-y}\int_{x=0}^{x=\sqrt{y}} f\left(x,y,z\right)dx\,dz\,dy$, (iii) $\int_{y=0}^{y=4}\int_{x=0}^{x=\sqrt{y}}\int_{z=0}^{z=4-y} f\left(x,y,z\right)dz\,dx\,dy$, (iv) $\int_{x=0}^{x=2}\int_{y=x^2}^{y=4}\int_{z=0}^{z=4-y} f\left(x,y,z\right)dz\,dy\,dx$, (v) $\int_{x=0}^{x=2}\int_{z=0}^{z=4-x^2}\int_{y=x^2}^{y=4-z} f\left(x,y,z\right)dy\,dz\,dx$

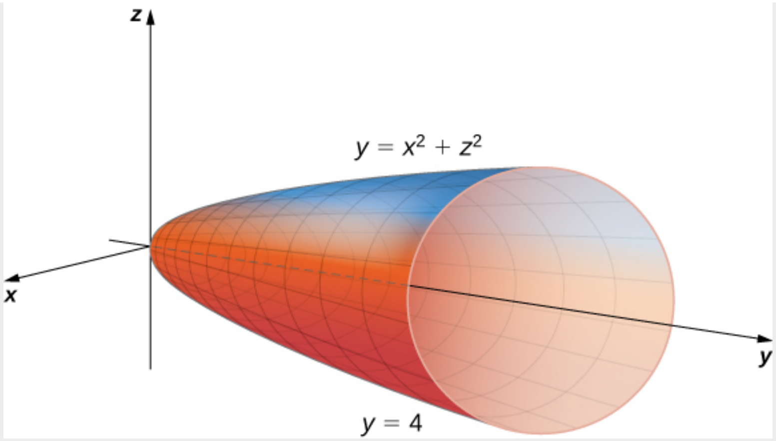
Hint

Follow the steps in the previous example, using the region E as $\left\{(x,y,z)|0\le z\le 4, 0\le y\le 4-z, 0\le x\le\sqrt{y}\right\}$, and describe and sketch the projections onto each of the three planes, five different times.

Example:
Exercise:

Problem:
Changing Integration Order and Coordinate Systems

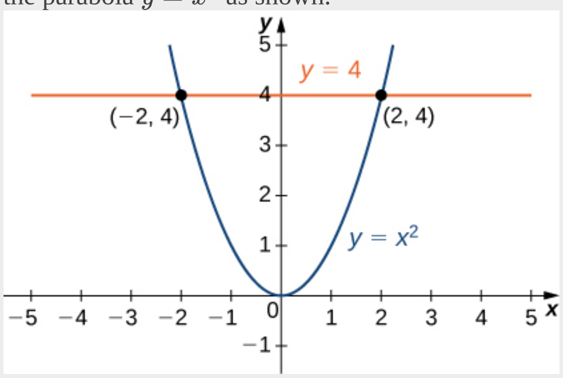
Evaluate the triple integral $\iiint\limits_E \sqrt{x^2+z^2}dV$, where E is the region bounded by the paraboloid $y=x^2+z^2$ ([link]) and the plane $y=4$.



Integrating a triple integral over a paraboloid.

Solution:

The projection of the solid region E onto the xy-plane is the region bounded above by $y=4$ and below by the parabola $y=x^2$ as shown.



Cross section in the xy-plane of the paraboloid in [link].

Thus, we have

Equation:

$$E=\left\{(x,y,z)\mid -2\le x\le 2, x^2\le y\le 4, -\sqrt{y-x^2}\le z\le\sqrt{y-x^2}\right\}.$$

The triple integral becomes

Equation:

$$\iiint_E \sqrt{x^2+z^2}\,dV=\int_{x=-2}^{x=2}\int_{y=x^2}^{y=4}\int_{z=-\sqrt{y-x^2}}^{z=\sqrt{y-x^2}}\sqrt{x^2+z^2}\,dz\,dy\,dx.$$

This expression is difficult to compute, so consider the projection of E onto the xz-plane. This is a circular disc $x^2+z^2 \le 4$. So we obtain

Equation:

$$\iiint_E \sqrt{x^2+z^2}\,dV = \int_{x=-2}^{x=2} \int_{y=x^2}^{y=4} \int_{z=-\sqrt{y-x^2}}^{z=\sqrt{y-x^2}} \sqrt{x^2+z^2}\,dz\,dy\,dx = \int_{x=-2}^{x=2} \int_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}} \int_{y=x^2+z^2}^{y=4} \sqrt{x^2+z^2}\,dy\,dz$$

Here the order of integration changes from being first with respect to z, then y, and then x to being first with respect to y, then to z, and then to x. It will soon be clear how this change can be beneficial for computation. We have

Equation:

$$\int_{x=-2}^{x=2} \int_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}} \int_{y=x^2+z^2}^{y=4} \sqrt{x^2+z^2}\,dy\,dz\,dx = \int_{x=-2}^{x=2} \int_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}} \left(4-x^2-z^2\right)\sqrt{x^2+z^2}\,dz\,dx.$$

Now use the polar substitution $x = r\cos\theta$, $z = r\sin\theta$, and $dz\,dx = r\,dr\,d\theta$ in the xz-plane. This is essentially the same thing as when we used polar coordinates in the xy-plane, except we are replacing y by z. Consequently the limits of integration change and we have, by using $r^2 = x^2+z^2$,

Equation:

$$\begin{aligned} \int_{x=-2}^{x=2} \int_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}} \left(4-x^2-z^2\right)\sqrt{x^2+z^2}\,dz\,dx &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} \left(4-r^2\right) r r\,dr\,d\theta \\ &= \int_0^{2\pi} \left[\frac{4r^3}{3} - \frac{r^5}{5}\Big|_0^2\right] d\theta = \int_0^{2\pi} \frac{64}{15}\,d\theta = \frac{128\pi}{15}. \end{aligned}$$

Average Value of a Function of Three Variables

Recall that we found the average value of a function of two variables by evaluating the double integral over a region on the plane and then dividing by the area of the region. Similarly, we can find the average value of a function in three variables by evaluating the triple integral over a solid region and then dividing by the volume of the solid.

Note:

Average Value of a Function of Three Variables

If $f(x,y,z)$ is integrable over a solid bounded region E with positive volume $V(E)$, then the average value of the function is

Equation:

$$f_{\text{ave}} = \frac{1}{V(E)} \iiint_E f(x,y,z)\,dV.$$

Note that the volume is $V(E) = \iiint\limits_E 1 dV$.

Example:
Exercise:

Problem:
Finding an Average Temperature

The temperature at a point (x, y, z) of a solid E bounded by the coordinate planes and the plane $x + y + z = 1$ is $T(x, y, z) = (xy + 8z + 20)^{\circ}$C. Find the average temperature over the solid.

Solution:

Use the theorem given above and the triple integral to find the numerator and the denominator. Then do the division. Notice that the plane $x + y + z = 1$ has intercepts $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. The region E looks like
Equation:

$$E = \{(x, y, z) | 0 \leq x \leq 1, 0 \leq y \leq 1 - x, 0 \leq z \leq 1 - x - y\}.$$

Hence the triple integral of the temperature is
Equation:

$$\iiint\limits_E f(x, y, z) dV = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x-y} (xy + 8z + 20) dz\, dy\, dx = \frac{147}{40}.$$

The volume evaluation is $V(E) = \iiint\limits_E 1 dV = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x-y} 1 dz\, dy\, dx = \frac{1}{6}$.

Hence the average value is $T_{\text{ave}} = \frac{147/40}{1/6} = \frac{6(147)}{40} = \frac{441}{20}$ degrees Celsius.

Note:
Exercise:

Problem:

Find the average value of the function $f(x, y, z) = xyz$ over the cube with sides of length 4 units in the first octant with one vertex at the origin and edges parallel to the coordinate axes.

Solution:

$f_{\text{ave}} = 8$

Hint

Follow the steps in the previous example.

Key Concepts

- To compute a triple integral we use Fubini's theorem, which states that if $f(x, y, z)$ is continuous on a rectangular box $B = [a, b] \times [c, d] \times [e, f]$, then
  Equation:

  $$\iiint_B f(x, y, z) dV = \int_e^f \int_c^d \int_a^b f(x, y, z) dx\, dy\, dz$$

  and is also equal to any of the other five possible orderings for the iterated triple integral.
- To compute the volume of a general solid bounded region E we use the triple integral
  Equation:

  $$V(E) = \iiint_E 1 dV.$$

- Interchanging the order of the iterated integrals does not change the answer. As a matter of fact, interchanging the order of integration can help simplify the computation.
- To compute the average value of a function over a general three-dimensional region, we use
  Equation:

  $$f_{\text{ave}} = \frac{1}{V(E)} \iiint_E f(x, y, z) dV.$$

Key Equations

- **Triple integral**
  $$\lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \Delta x \Delta y \Delta z = \iiint_B f(x, y, z) dV$$

In the following exercises, evaluate the triple integrals over the rectangular solid box B.

Exercise:

Problem: $\iiint_B (2x + 3y^2 + 4z^3) dV$, where $B = \{(x, y, z) | 0 \le x \le 1, 0 \le y \le 2, 0 \le z \le 3\}$

Solution:

192

Exercise:

Problem: $\iiint_B (xy + yz + xz) dV$, where $B = \{(x, y, z) | 1 \le x \le 2, 0 \le y \le 2, 1 \le z \le 3\}$

Exercise:

Problem: $\iiint\limits_B (x\cos y + z)dV$, where $B = \{(x,y,z)|0 \le x \le 1, 0 \le y \le \pi, -1 \le z \le 1\}$

Solution:

0

Exercise:

Problem: $\iiint\limits_B \left(z\sin x + y^2\right)dV$, where $B = \{(x,y,z)|0 \le x \le \pi, 0 \le y \le 1, -1 \le z \le 2\}$

In the following exercises, change the order of integration by integrating first with respect to z, then x, then y.

Exercise:

Problem: $\int\limits_0^1 \int\limits_1^2 \int\limits_2^3 \left(x^2 + \ln y + z\right)dx\,dy\,dz$

Solution:

$$\int\limits_1^2 \int\limits_2^3 \int\limits_0^1 \left(x^2 + \ln y + z\right)dz\,dx\,dy = \frac{35}{6} + 2\ln 2$$

Exercise:

Problem: $\int\limits_0^1 \int\limits_{-1}^1 \int\limits_0^3 (ze^x + 2y)dx\,dy\,dz$

Exercise:

Problem: $\int\limits_{-1}^2 \int\limits_1^3 \int\limits_0^4 \left(x^2 z + \frac{1}{y}\right)dx\,dy\,dz$

Solution:

$$\int\limits_1^3 \int\limits_0^4 \int\limits_{-1}^2 \left(x^2 z + \frac{1}{y}\right)dz\,dx\,dy = 64 + 12\ln 3$$

Exercise:

Problem: $\int\limits_1^2 \int\limits_{-2}^{-1} \int\limits_0^1 \frac{x+y}{z}\,dx\,dy\,dz$

Exercise:

Problem:

Let $F, G,$ and H be continuous functions on $[a, b]$, $[c, d]$, and $[e, f]$, respectively, where $a, b, c, d, e,$ and f are real numbers such that $a < b, c < d,$ and $e < f$. Show that

Equation:

$$\int_a^b \int_c^d \int_e^f F(x)G(y)H(z)dz\,dy\,dx = \left(\int_a^b F(x)dx\right)\left(\int_c^d G(y)dy\right)\left(\int_e^f H(z)dz\right).$$

Exercise:

Problem:

Let $F, G,$ and H be differential functions on $[a, b]$, $[c, d]$, and $[e, f]$, respectively, where $a, b, c, d, e,$ and f are real numbers such that $a < b, c < d,$ and $e < f$. Show that

Equation:

$$\int_a^b \int_c^d \int_e^f F'(x)G'(y)H'(z)dz\,dy\,dx = [F(b) - F(a)]\,[G(d) - G(c)]\,[H(f) - H(e)].$$

In the following exercises, evaluate the triple integrals over the bounded region $E = \{(x, y, z)|a \le x \le b, h_1(x) \le y \le h_2(x), e \le z \le f\}$.

Exercise:

Problem: $\iiint\limits_E (2x + 5y + 7z)dV$, where $E = \{(x, y, z)|0 \le x \le 1, 0 \le y \le -x + 1, 1 \le z \le 2\}$

Solution:

$\frac{77}{12}$

Exercise:

Problem: $\iiint\limits_E (y \ln x + z)dV$, where $E = \{(x, y, z)|1 \le x \le e, 0 \le y \le \ln x, 0 \le z \le 1\}$

Exercise:

Problem: $\iiint\limits_E (\sin x + \sin y)dV$, where $E = \left\{(x, y, z)|0 \le x \le \frac{\pi}{2}, -\cos x \le y \le \cos x, -1 \le z \le 1\right\}$

Solution:

2

Exercise:

Problem: $\iiint\limits_E (xy + yz + xz)dV$, where $E = \left\{(x, y, z)|0 \le x \le 1, -x^2 \le y \le x^2, 0 \le z \le 1\right\}$

In the following exercises, evaluate the triple integrals over the indicated bounded region E.

Exercise:

Problem: $\iiint\limits_E (x + 2yz)dV$, where $E = \{(x, y, z)|0 \le x \le 1, 0 \le y \le x, 0 \le z \le 5 - x - y\}$

Solution:

$\frac{439}{120}$

Exercise:

Problem: $\iiint\limits_E \left(x^3 + y^3 + z^3\right)dV$, where $E = \{(x, y, z)|0 \le x \le 2, 0 \le y \le 2x, 0 \le z \le 4 - x - y\}$

Exercise:

Problem:

$\iiint\limits_E y\,dV$, where $E = \left\{(x, y, z)\middle| -1 \le x \le 1, -\sqrt{1 - x^2} \le y \le \sqrt{1 - x^2}, 0 \le z \le 1 - x^2 - y^2\right\}$

Solution:

0

Exercise:

Problem:

$\iiint\limits_E x\,dV$, where $E = \left\{(x, y, z)\middle| -2 \le x \le 2, -4\sqrt{1 - x^2} \le y \le \sqrt{4 - x^2}, 0 \le z \le 4 - x^2 - y^2\right\}$

In the following exercises, evaluate the triple integrals over the bounded region E of the form $E = \{(x, y, z)|g_1(y) \le x \le g_2(y), c \le y \le d, e \le z \le f\}$.

Exercise:

Problem: $\iiint\limits_E x^2dV$, where $E = \left\{(x, y, z)|1 - y^2 \le x \le y^2 - 1, -1 \le y \le 1, 1 \le z \le 2\right\}$

Solution:

$-\frac{64}{105}$

Exercise:

Problem: $\iiint\limits_E (\sin x + y)dV$, where $E = \left\{(x, y, z)\middle| -y^4 \le x \le y^4, 0 \le y \le 2, 0 \le z \le 4\right\}$

Exercise:

Problem: $\iiint\limits_E (x - yz)dV$, where $E = \left\{(x, y, z)\middle| -y^6 \le x \le \sqrt{y}, 0 \le y \le 1x, -1 \le z \le 1\right\}$

Solution:

$\frac{11}{26}$

Exercise:

Problem: $\iiint\limits_E zdV$, where $E = \left\{(x,y,z)|2-2y \le x \le 2+\sqrt{y}, 0 \le y \le 1x, 2 \le z \le 3\right\}$

In the following exercises, evaluate the triple integrals over the bounded region

Equation:

$$E = \{(x,y,z)|g_1(y) \le x \le g_2(y), c \le y \le d, u_1(x,y) \le z \le u_2(x,y)\}.$$

Exercise:

Problem: $\iiint\limits_E zdV$, where $E = \left\{(x,y,z)\middle| -y \le x \le y, 0 \le y \le 1, 0 \le z \le 1-x^4-y^4\right\}$

Solution:

$\frac{113}{450}$

Exercise:

Problem: $\iiint\limits_E (xz+1)dV$, where $E = \left\{(x,y,z)\middle|0 \le x \le \sqrt{y}, 0 \le y \le 2, 0 \le z \le 1-x^2-y^2\right\}$

Exercise:

Problem:

$\iiint\limits_E (x-z)dV$, where $E = \left\{(x,y,z)\middle| -\sqrt{1-y^2} \le x \le y, 0 \le y \le \frac{1}{2}x, 0 \le z \le 1-x^2-y^2\right\}$

Solution:

$\frac{1}{160}\left(6\sqrt{3}-41\right)$

Exercise:

Problem: $\iiint\limits_E (x+y)dV$, where $E = \left\{(x,y,z)\middle|0 \le x \le \sqrt{1-y^2}, 0 \le y \le 1x, 0 \le z \le 1-x\right\}$

In the following exercises, evaluate the triple integrals over the bounded region

$E = \{(x,y,z)|\,(x,y) \in D, u_1(x,y)x \le z \le u_2(x,y)\}$, where D is the projection of E onto the xy-plane.

Exercise:

Problem: $\iint\limits_D \left(\int_1^2 (x+z)dz\right)dA$, where $D = \left\{(x,y)|x^2+y^2 \le 1\right\}$

Solution:

$\frac{3\pi}{2}$

Exercise:

Problem: $\iint\limits_D \left(\int\limits_1^3 x\,(z+1)dz \right) dA$, where $D = \left\{ (x,y) \middle| x^2 - y^2 \geq 1, x \leq \sqrt{5} \right\}$

Exercise:

Problem: $\iint\limits_D \left(\int\limits_0^{10-x-y} (x+2z)dz \right) dA$, where $D = \{(x,y)|y \geq 0, x \geq 0, x+y \leq 10\}$

Solution:

1250

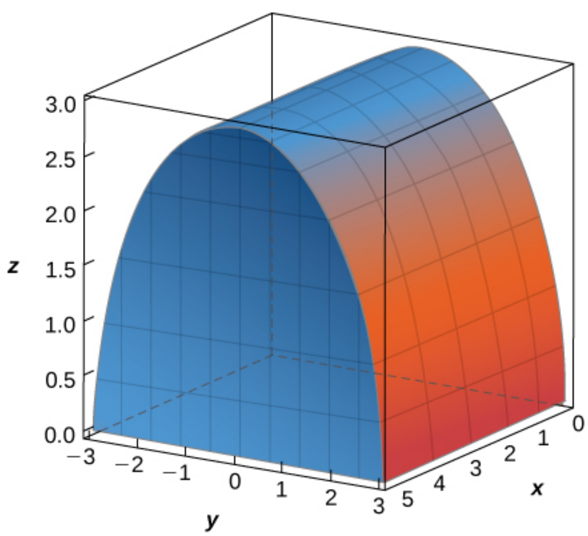
Exercise:

Problem: $\iint\limits_D \left(\int\limits_0^{4x^2+4y^2} y\,dz \right) dA$, where $D = \left\{ (x,y) \middle| x^2 + y^2 \leq 4, y \geq 1, x \geq 0 \right\}$

Exercise:

Problem:

The solid E bounded by $y^2 + z^2 = 9$, $z = 0$, and $x = 5$ is shown in the following figure. Evaluate the integral $\iiint\limits_E z\,dV$ by integrating first with respect to z, then y, and then x.



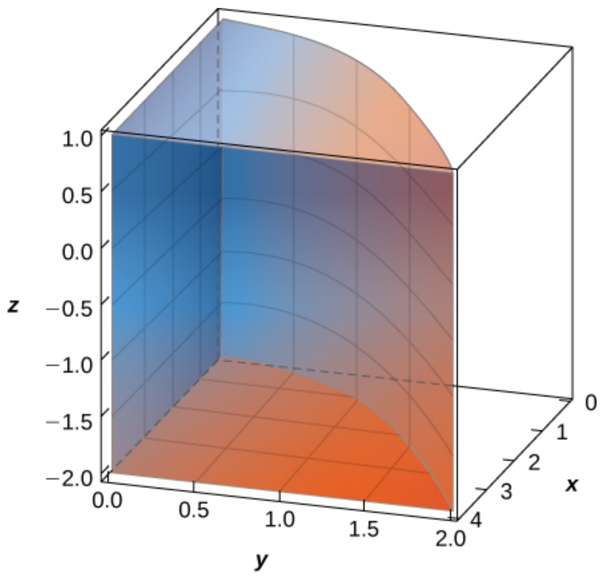
Solution:

$$\int_0^5 \int_{-3}^3 \int_0^{\sqrt{9-y^2}} z\,dz\,dy\,dx = 90$$

Exercise:

Problem:

The solid E bounded by $y = \sqrt{x}$, $x = 4$, $y = 0$, and $z = 1$ is given in the following figure. Evaluate the integral $\iiint\limits_E xyz\,dV$ by integrating first with respect to x, then y, and then z.



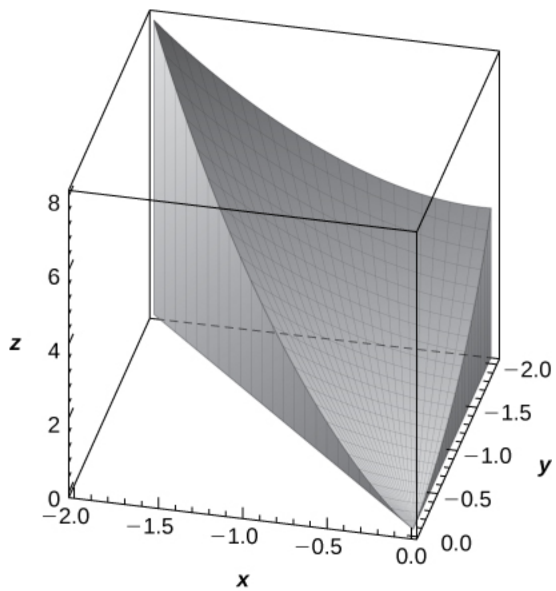
Exercise:

Problem:

[T] The volume of a solid E is given by the integral $\int_{-2}^0 \int_x^0 \int_0^{x^2+y^2} dz\,dy\,dx$. Use a computer algebra system (CAS) to graph E and find its volume. Round your answer to two decimal places.

Solution:

$V = 5.33$



Exercise:

Problem:

[T] The volume of a solid E is given by the integral $\int_{-1}^{0}\int_{-x^2}^{0}\int_{0}^{1+\sqrt{x^2+y^2}} dz\,dy\,dx$. Use a CAS to graph E and find its volume V. Round your answer to two decimal places.

In the following exercises, use two circular permutations of the variables $x, y,$ and z to write new integrals whose values equal the value of the original integral. A circular permutation of $x, y,$ and z is the arrangement of the numbers in one of the following orders: $y, z,$ and x or $z, x,$ and y.

Exercise:

Problem: $\int_{0}^{1}\int_{1}^{3}\int_{2}^{4}\left(x^2z^2+1\right)dx\,dy\,dz$

Solution:

$\int_{0}^{1}\int_{1}^{3}\int_{2}^{4}\left(y^2z^2+1\right)dz\,dx\,dy;\ \int_{0}^{1}\int_{1}^{3}\int_{2}^{4}\left(x^2y^2+1\right)dy\,dz\,dx$

Exercise:

Problem: $\int_{1}^{3}\int_{0}^{1}\int_{0}^{-x+1}(2x+5y+7z)dy\,dx\,dz$

Exercise:

Problem: $\int_{0}^{1}\int_{-y}^{y}\int_{0}^{1-x^4-y^4} \ln x \, dz \, dx \, dy$

Exercise:

Problem: $\int_{-1}^{1}\int_{0}^{1}\int_{-y^6}^{\sqrt{y}} (x+yz) dx \, dy \, dz$

Exercise:

Problem:

Set up the integral that gives the volume of the solid E bounded by $y^2 = x^2 + z^2$ and $y = a^2$, where $a > 0$.

Solution:

$$V = \int_{-a}^{a}\int_{-\sqrt{a^2-z^2}}^{\sqrt{a^2-z^2}}\int_{\sqrt{x^2+z^2}}^{a^2} dy \, dx \, dz$$

Exercise:

Problem:

Set up the integral that gives the volume of the solid E bounded by $x = y^2 + z^2$ and $x = a^2$, where $a > 0$.

Exercise:

Problem:

Find the average value of the function $f(x, y, z) = x + y + z$ over the parallelepiped determined by $x = 0, x = 1, y = 0, y = 3, z = 0,$ and $z = 5$.

Solution:

$\frac{9}{2}$

Exercise:

Problem:

Find the average value of the function $f(x, y, z) = xyz$ over the solid $E = [0, 1] \times [0, 1] \times [0, 1]$ situated in the first octant.

Exercise:

Problem:

Find the volume of the solid E that lies under the plane $x + y + z = 9$ and whose projection onto the xy-plane is bounded by $x = \sqrt{y - 1}, x = 0,$ and $x + y = 7$.

Solution:

$\frac{156}{5}$

Exercise:

Problem:

Find the volume of the solid E that lies under the plane $2x+y+z=8$ and whose projection onto the xy-plane is bounded by $x=\sin^{-1}y, y=0$, and $x=\frac{\pi}{2}$.

Exercise:

Problem:

Consider the pyramid with the base in the xy-plane of $[-2,2]\times[-2,2]$ and the vertex at the point $(0,0,8)$.

a. Show that the equations of the planes of the lateral faces of the pyramid are $4y+z=8$, $4y-z=-8$, $4x+z=8$, and $-4x+z=8$.
b. Find the volume of the pyramid.

Solution:

a. Answers may vary; b. $\frac{128}{3}$

Exercise:

Problem:

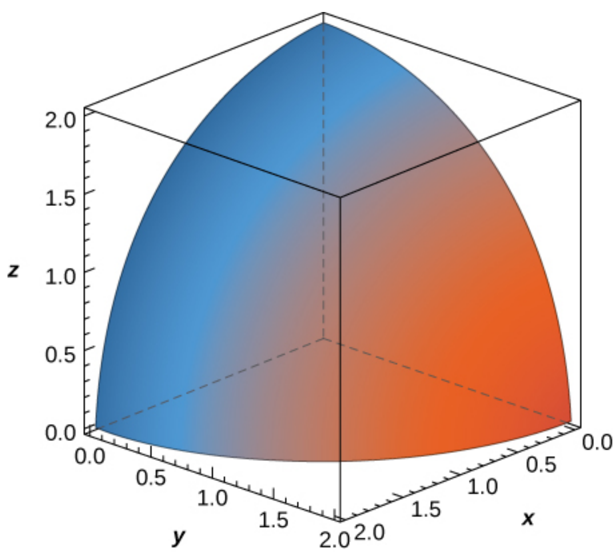
Consider the pyramid with the base in the xy-plane of $[-3,3]\times[-3,3]$ and the vertex at the point $(0,0,9)$.

a. Show that the equations of the planes of the side faces of the pyramid are $3y+z=9$, $3y+z=9$, $y=0$ and $x=0$.
b. Find the volume of the pyramid.

Exercise:

Problem:

The solid E bounded by the sphere of equation $x^2+y^2+z^2=r^2$ with $r>0$ and located in the first octant is represented in the following figure.



a. Write the triple integral that gives the volume of E by integrating first with respect to z, then with y, and then with x.
b. Rewrite the integral in part a. as an equivalent integral in five other orders.

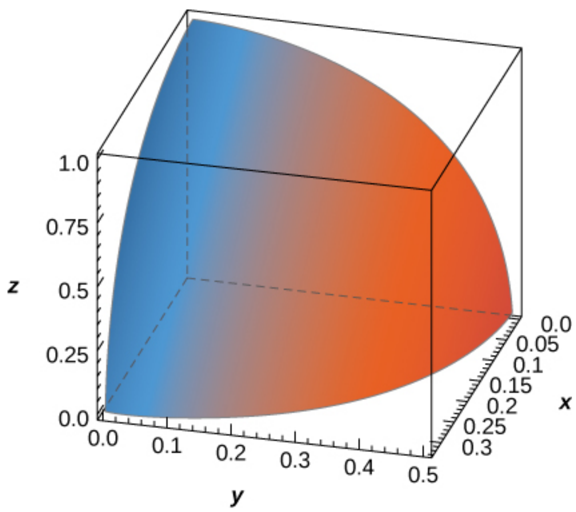
Solution:

a. $\int_0^4 \int_0^{\sqrt{r^2-x^2}} \int_0^{\sqrt{r^2-x^2-y^2}} dz\, dy\, dx$; b. $\int_0^2 \int_0^{\sqrt{r^2-y^2}} \int_0^{\sqrt{r^2-x^2-y^2}} dz\, dx\, dy$, $\int_0^r \int_0^{\sqrt{r^2-z^2}} \int_0^{\sqrt{r^2-x^2-z^2}} dy\, dx\, dz$, $\int_0^r \int_0^{\sqrt{r^2-x^2}} \int_0^{\sqrt{r^2-x^2-z^2}} dy\, dz\, dx$, $\int_0^r \int_0^{\sqrt{r^2-z^2}} \int_0^{\sqrt{r^2-y^2-z^2}} dx\, dy\, dz$, $\int_0^r \int_0^{\sqrt{r^2-y^2}} \int_0^{\sqrt{r^2-y^2-z^2}} dx\, dz\, dy$

Exercise:

Problem:

The solid E bounded by the equation $9x^2 + 4y^2 + z^2 = 1$ and located in the first octant is represented in the following figure.



a. Write the triple integral that gives the volume of E by integrating first with respect to z, then with y, and then with x.
b. Rewrite the integral in part a. as an equivalent integral in five other orders.

Exercise:

Problem:

Find the volume of the prism with vertices $(0, 0, 0), (2, 0, 0), (2, 3, 0), (0, 3, 0), (0, 0, 1), \text{ and } (2, 0, 1)$.

Solution:

3

Exercise:

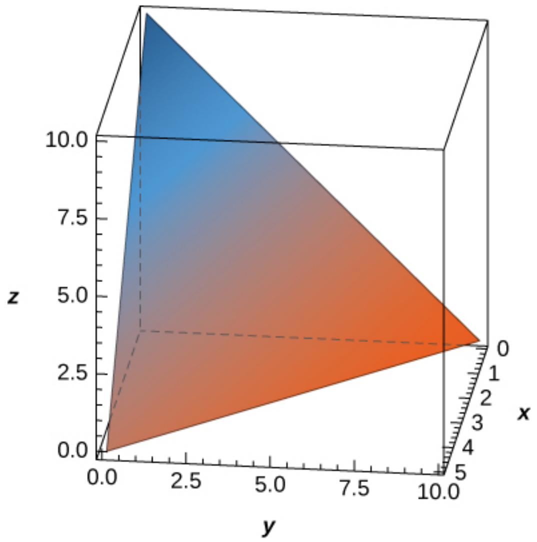
Problem:

Find the volume of the prism with vertices $(0, 0, 0), (4, 0, 0), (4, 6, 0), (0, 6, 0), (0, 0, 1), \text{ and } (4, 0, 1)$.

Exercise:

Problem:

The solid E bounded by $z = 10 - 2x - y$ and situated in the first octant is given in the following figure. Find the volume of the solid.



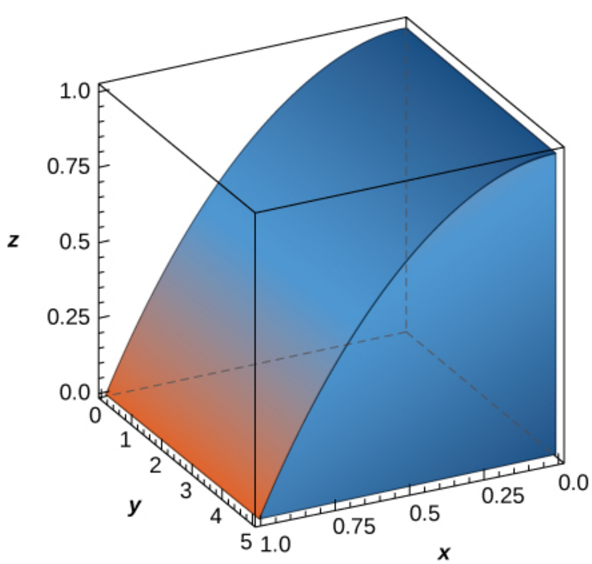
Solution:

$\frac{250}{3}$

Exercise:

Problem:

The solid E bounded by $z = 1 - x^2$ and situated in the first octant is given in the following figure. Find the volume of the solid.



Exercise:

Problem:

The midpoint rule for the triple integral $\iiint\limits_B f(x,y,z)dV$ over the rectangular solid box B is a generalization of the midpoint rule for double integrals. The region B is divided into subboxes of equal sizes and the integral is approximated by the triple Riemann sum $\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(\overline{x_i},\overline{y_j},\overline{z_k})\Delta V$, where $(\overline{x_i},\overline{y_j},\overline{z_k})$ is the center of the box B_{ijk} and ΔV is the volume of each subbox. Apply the midpoint rule to approximate $\iiint\limits_B x^2 dV$ over the solid $B=\{(x,y,z)|0\le x\le 1, 0\le y\le 1, 0\le z\le 1\}$ by using a partition of eight cubes of equal size. Round your answer to three decimal places.

Solution:

$\frac{5}{16}\approx 0.313$

Exercise:

Problem: **[T]**

a. Apply the midpoint rule to approximate $\iiint\limits_B e^{-x^2}dV$ over the solid $B=\{(x,y,z)|0\le x\le 1, 0\le y\le 1, 0\le z\le 1\}$ by using a partition of eight cubes of equal size. Round your answer to three decimal places.
b. Use a CAS to improve the above integral approximation in the case of a partition of n^3 cubes of equal size, where $n=3,4,\ldots,10$.

Exercise:

Problem:

Suppose that the temperature in degrees Celsius at a point (x,y,z) of a solid E bounded by the coordinate planes and $x+y+z=5$ is $T(x,y,z)=xz+5z+10$. Find the average temperature over the solid.

Solution:

$\frac{35}{2}$

Exercise:

Problem:

Suppose that the temperature in degrees Fahrenheit at a point (x,y,z) of a solid E bounded by the coordinate planes and $x+y+z=5$ is $T(x,y,z)=x+y+xy$. Find the average temperature over the solid.

Exercise:

Problem:

Show that the volume of a right square pyramid of height h and side length a is $v=\frac{ha^2}{3}$ by using triple integrals.

Exercise:

Problem:

Show that the volume of a regular right hexagonal prism of edge length a is $\frac{3a^3\sqrt{3}}{2}$ by using triple integrals.

Exercise:

Problem:

Show that the volume of a regular right hexagonal pyramid of edge length a is $\frac{a^3\sqrt{3}}{2}$ by using triple integrals.

Exercise:

Problem:

If the charge density at an arbitrary point (x, y, z) of a solid E is given by the function $\rho(x, y, z)$, then the total charge inside the solid is defined as the triple integral $\iiint\limits_E \rho(x, y, z)dV$. Assume that the charge density of the solid E enclosed by the paraboloids $x = 5 - y^2 - z^2$ and $x = y^2 + z^2 - 5$ is equal to the distance from an arbitrary point of E to the origin. Set up the integral that gives the total charge inside the solid E.

Glossary

triple integral
: the triple integral of a continuous function $f(x, y, z)$ over a rectangular solid box B is the limit of a Riemann sum for a function of three variables, if this limit exists

Triple Integrals in Cylindrical and Spherical Coordinates

- Evaluate a triple integral by changing to cylindrical coordinates.
- Evaluate a triple integral by changing to spherical coordinates.

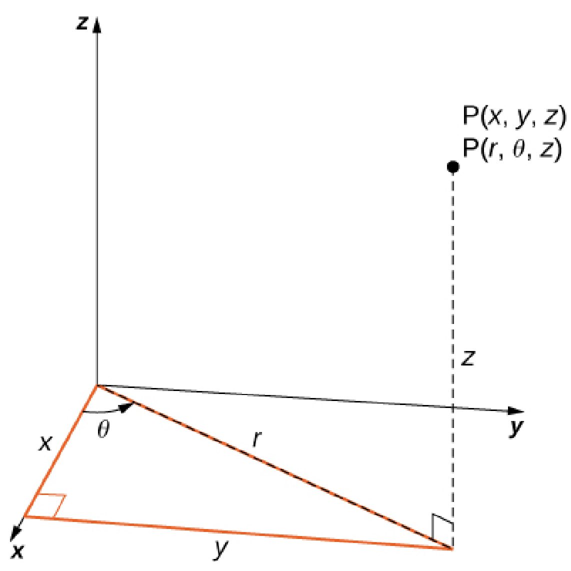
Earlier in this chapter we showed how to convert a double integral in rectangular coordinates into a double integral in polar coordinates in order to deal more conveniently with problems involving circular symmetry. A similar situation occurs with triple integrals, but here we need to distinguish between cylindrical symmetry and spherical symmetry. In this section we convert triple integrals in rectangular coordinates into a triple integral in either cylindrical or spherical coordinates.

Also recall the chapter opener, which showed the opera house l'Hemisphèric in Valencia, Spain. It has four sections with one of the sections being a theater in a five-story-high sphere (ball) under an oval roof as long as a football field. Inside is an IMAX screen that changes the sphere into a planetarium with a sky full of 9000 twinkling stars. Using triple integrals in spherical coordinates, we can find the volumes of different geometric shapes like these.

Review of Cylindrical Coordinates

As we have seen earlier, in two-dimensional space $\mathbb{R}^2$, a point with rectangular coordinates (x, y) can be identified with (r, θ) in polar coordinates and vice versa, where $x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2 + y^2$ and $\tan\theta = \left(\frac{y}{x}\right)$ are the relationships between the variables.

In three-dimensional space $\mathbb{R}^3$, a point with rectangular coordinates (x, y, z) can be identified with cylindrical coordinates (r, θ, z) and vice versa. We can use these same conversion relationships, adding z as the vertical distance to the point from the xy-plane as shown in the following figure.



Cylindrical coordinates are similar to polar coordinates with a vertical z coordinate added.

To convert from rectangular to cylindrical coordinates, we use the conversion $x = r\cos\theta$ and $y = r\sin\theta$. To convert from cylindrical to rectangular coordinates, we use $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$. The z-coordinate remains the same in both cases.

In the two-dimensional plane with a rectangular coordinate system, when we say $x = k$ (constant) we mean an unbounded vertical line parallel to the y-axis and when $y = l$ (constant) we mean an unbounded horizontal line parallel to the x-axis. With the polar coordinate system, when we say $r = c$ (constant), we mean a circle of radius c units and when $\theta = \alpha$ (constant) we mean an infinite ray making an angle α with the positive x-axis.

Similarly, in three-dimensional space with rectangular coordinates (x, y, z), the equations $x = k, y = l$, and $z = m$, where k, l, and m are constants, represent unbounded planes parallel to the yz-plane, xz-plane and xy-plane, respectively. With cylindrical coordinates (r, θ, z), by $r = c, \theta = \alpha$, and $z = m$, where c, α, and m are constants, we mean an unbounded vertical cylinder with the z-axis as its radial axis; a plane making a constant angle α with the xy-plane; and an unbounded horizontal plane parallel to the xy-plane, respectively. This means that the circular cylinder $x^2 + y^2 = c^2$ in rectangular coordinates can be represented simply as $r = c$ in cylindrical coordinates. (Refer to Cylindrical and Spherical Coordinates for more review.)

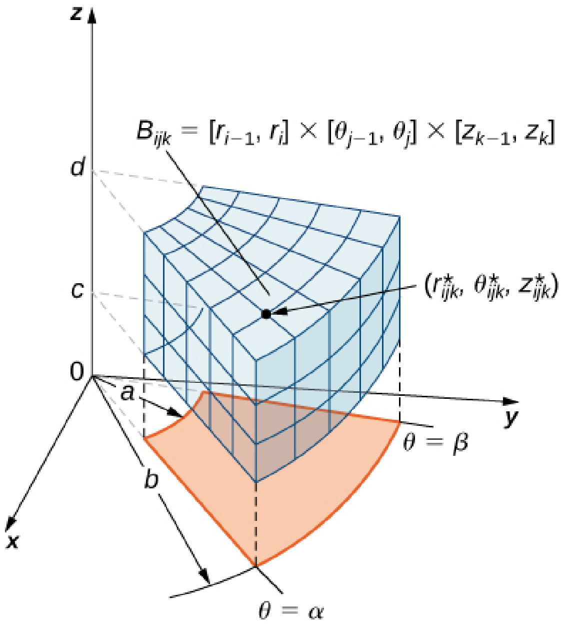
Integration in Cylindrical Coordinates

Triple integrals can often be more readily evaluated by using cylindrical coordinates instead of rectangular coordinates. Some common equations of surfaces in rectangular coordinates along with corresponding equations in cylindrical coordinates are listed in [link]. These equations will become handy as we proceed with solving problems using triple integrals.

	Circular cylinder	**Circular cone**	**Sphere**	**Paraboloid**
Rectangular	$x^2 + y^2 = c^2$	$z^2 = c^2\left(x^2 + y^2\right)$	$x^2 + y^2 + z^2 = c^2$	$z = c\left(x^2 + y^2\right)$
Cylindrical	$r = c$	$z = cr$	$r^2 + z^2 = c^2$	$z = cr^2$

Equations of Some Common Shapes

As before, we start with the simplest bounded region B in $\mathbb{R}^3$, to describe in cylindrical coordinates, in the form of a cylindrical box, $B = \{(r, \theta, z) | a \le r \le b, \alpha \le \theta \le \beta, c \le z \le d\}$ ([link]). Suppose we divide each interval into l, m and n subdivisions such that $\Delta r = \frac{b-a}{l}$, $\Delta\theta = \frac{\beta-\alpha}{m}$, and $\Delta z = \frac{d-c}{n}$. Then we can state the following definition for a triple integral in cylindrical coordinates.



A cylindrical box B described by cylindrical coordinates.

Note:
Definition
Consider the cylindrical box (expressed in cylindrical coordinates)
Equation:

$$B = \{(r, \theta, z) | a \le r \le b, \alpha \le \theta \le \beta, c \le z \le d\}.$$

If the function $f(r, \theta, z)$ is continuous on B and if $(r^*_{ijk}, \theta^*_{ijk}, z^*_{ijk})$ is any sample point in the cylindrical subbox $B_{ijk} = [r_{i-1}, r_i] \times [\theta_{j-1}, \theta_j] \times [z_{k-1}, z_k]$ ([link]), then we can define the **triple integral in cylindrical coordinates** as the limit of a triple Riemann sum, provided the following limit exists:
Equation:

$$\lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(r^*_{ijk}, \theta^*_{ijk}, z^*_{ijk}) r^*_{ijk} \Delta r \Delta\theta \Delta z.$$

Note that if $g(x, y, z)$ is the function in rectangular coordinates and the box B is expressed in rectangular coordinates, then the triple integral $\iiint\limits_B g(x, y, z) dV$ is equal to the triple integral $\iiint\limits_B g(r\cos\theta, r\sin\theta, z) r\, dr\, d\theta\, dz$ and we have

Equation:

$$\iiint_B g(x,y,z)dV = \iiint_B g(r\cos\theta, r\sin\theta, z)r\,dr\,d\theta\,dz = \iiint_B f(r,\theta,z)r\,dr\,d\theta\,dz.$$

As mentioned in the preceding section, all the properties of a double integral work well in triple integrals, whether in rectangular coordinates or cylindrical coordinates. They also hold for iterated integrals. To reiterate, in cylindrical coordinates, Fubini's theorem takes the following form:

Note:
Fubini's Theorem in Cylindrical Coordinates
Suppose that $g(x,y,z)$ is continuous on a rectangular box B, which when described in cylindrical coordinates looks like $B = \{(r,\theta,z) | a \le r \le b, \alpha \le \theta \le \beta, c \le z \le d\}$.
Then $g(x,y,z) = g(r\cos\theta, r\sin\theta, z) = f(r,\theta,z)$ and
Equation:

$$\iiint_B g(x,y,z)dV = \int_c^d \int_\alpha^\beta \int_a^b f(r,\theta,z)r\,dr\,d\theta\,dz.$$

The iterated integral may be replaced equivalently by any one of the other five iterated integrals obtained by integrating with respect to the three variables in other orders.

Cylindrical coordinate systems work well for solids that are symmetric around an axis, such as cylinders and cones. Let us look at some examples before we define the triple integral in cylindrical coordinates on general cylindrical regions.

Example:
Exercise:

Problem:
Evaluating a Triple Integral over a Cylindrical Box

Evaluate the triple integral $\iiint_B (zr\sin\theta)r\,dr\,d\theta\,dz$ where the cylindrical box B is

$B = \{(r,\theta,z) | 0 \le r \le 2, 0 \le \theta \le \pi/2, 0 \le z \le 4\}$.

Solution:

As stated in Fubini's theorem, we can write the triple integral as the iterated integral
Equation:

$$\iiint_B (zr\sin\theta)r\,dr\,d\theta\,dz = \int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=2} \int_{z=0}^{z=4} (zr\sin\theta)r\,dz\,dr\,d\theta.$$

The evaluation of the iterated integral is straightforward. Each variable in the integral is independent of the others, so we can integrate each variable separately and multiply the results together. This makes the computation much easier:

Equation:

$$\int_{\theta=0}^{\theta=\pi/2}\int_{r=0}^{r=2}\int_{z=0}^{z=4}(zr\sin\theta)r\,dz\,dr\,d\theta$$
$$=\left(\int_0^{\pi/2}\sin\theta\,d\theta\right)\left(\int_0^2 r^2dr\right)\left(\int_0^4 z\,dz\right)=\left(-\cos\theta|_0^{\pi/2}\right)\left(\frac{r^3}{3}\Big|_0^2\right)\left(\frac{z^2}{2}\Big|_0^4\right)=\frac{64}{3}.$$

Note:

Exercise:

Problem: Evaluate the triple integral $\int\limits_{\theta=0}^{\theta=\pi}\int\limits_{r=0}^{r=1}\int\limits_{z=0}^{z=4} rz\sin\theta r\,dz\,dr\,d\theta.$

Solution:

8

Hint

Follow the same steps as in the previous example.

If the cylindrical region over which we have to integrate is a general solid, we look at the projections onto the coordinate planes. Hence the triple integral of a continuous function $f(r,\theta,z)$ over a general solid region $E=\{(r,\theta,z)|\,(r,\theta)\in D, u_1(r,\theta)\le z\le u_2(r,\theta)\}$ in $\mathbb{R}^3$, where D is the projection of E onto the $r\theta$-plane, is

Equation:

$$\iiint\limits_E f(r,\theta,z)r\,dr\,d\theta\,dz=\iint\limits_D\left[\int\limits_{u_1(r,\theta)}^{u_2(r,\theta)} f(r,\theta,z)dz\right]r\,dr\,d\theta.$$

In particular, if $D=\{(r,\theta)|g_1(\theta)\le r\le g_2(\theta),\alpha\le\theta\le\beta\}$, then we have

Equation:

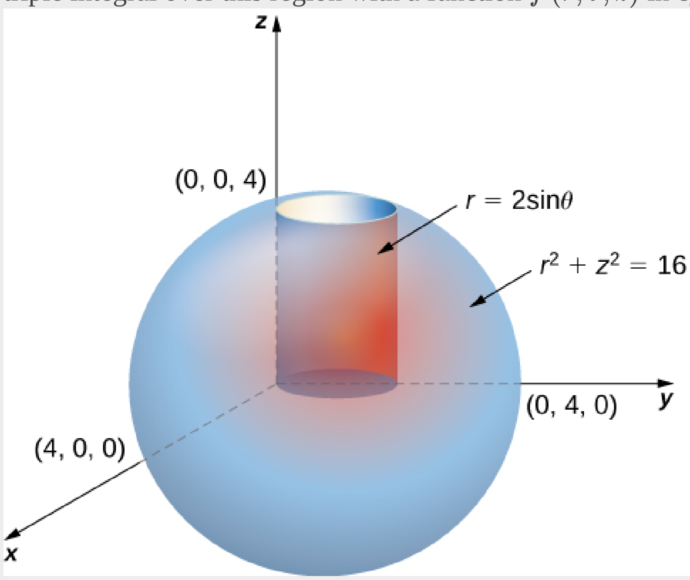
$$\iiint\limits_E f(r,\theta,z)r\,dr\,d\theta=\int\limits_{\theta=\alpha}^{\theta=\beta}\int\limits_{r=g_1(\theta)}^{r=g_2(\theta)}\int\limits_{z=u_1(r,\theta)}^{z=u_2(r,\theta)} f(r,\theta,z)r\,dz\,dr\,d\theta.$$

Similar formulas exist for projections onto the other coordinate planes. We can use polar coordinates in those planes if necessary.

Example:
Exercise:

Problem:
Setting up a Triple Integral in Cylindrical Coordinates over a General Region

Consider the region E inside the right circular cylinder with equation $r = 2\sin\theta$, bounded below by the $r\theta$-plane and bounded above by the sphere with radius 4 centered at the origin ([link]). Set up a triple integral over this region with a function $f(r, \theta, z)$ in cylindrical coordinates.



Setting up a triple integral in cylindrical coordinates over a cylindrical region.

Solution:

First, identify that the equation for the sphere is $r^2 + z^2 = 16$. We can see that the limits for z are from 0 to $z = \sqrt{16 - r^2}$. Then the limits for r are from 0 to $r = 2\sin\theta$. Finally, the limits for θ are from 0 to π. Hence the region is
Equation:

$$E = \left\{(r, \theta, z) | 0 \le \theta \le \pi, 0 \le r \le 2\sin\theta, 0 \le z \le \sqrt{16 - r^2}\right\}.$$

Therefore, the triple integral is
Equation:

$$\iiint_E f(r,\theta,z) r \, dz \, dr \, d\theta = \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=2\sin\theta} \int_{z=0}^{z=\sqrt{16-r^2}} f(r,\theta,z) r \, dz \, dr \, d\theta.$$

Note:
Exercise:

Problem:

Consider the region E inside the right circular cylinder with equation $r = 2\sin\theta$, bounded below by the $r\theta$-plane and bounded above by $z = 4 - y$. Set up a triple integral with a function $f(r,\theta,z)$ in cylindrical coordinates.

Solution:

$$\iiint_E f(r,\theta,z) r \, dz \, dr \, d\theta = \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=2\sin\theta} \int_{z=0}^{z=4-r\sin\theta} f(r,\theta,z) r \, dz \, dr \, d\theta.$$

Hint

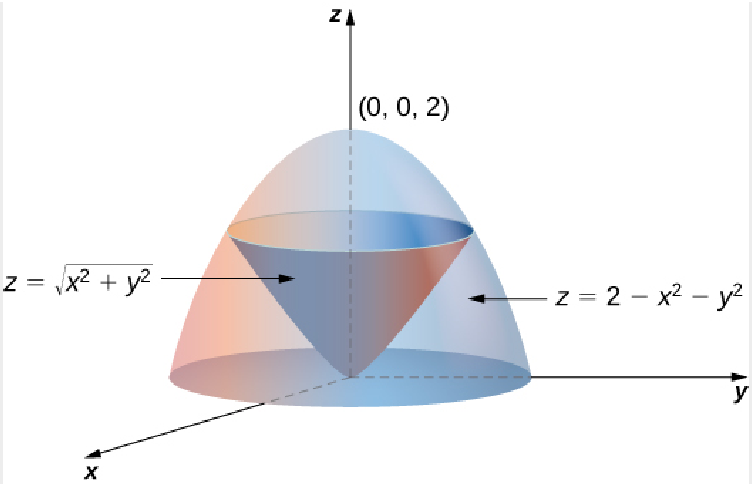
Analyze the region, and draw a sketch.

Example:
Exercise:

Problem:
Setting up a Triple Integral in Two Ways

Let E be the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the paraboloid $z = 2 - x^2 - y^2$. ([link]). Set up a triple integral in cylindrical coordinates to find the volume of the region, using the following orders of integration:

a. $dz \, dr \, d\theta$
b. $dr \, dz \, d\theta$.



Setting up a triple integral in cylindrical coordinates over a conical region.

Solution:

a. The cone is of radius 1 where it meets the paraboloid. Since $z = 2 - x^2 - y^2 = 2 - r^2$ and $z = \sqrt{x^2 + y^2} = r$ (assuming r is nonnegative), we have $2 - r^2 = r$. Solving, we have $r^2 + r - 2 = (r + 2)(r - 1) = 0$. Since $r \geq 0$, we have $r = 1$. Therefore $z = 1$. So the intersection of these two surfaces is a circle of radius 1 in the plane $z = 1$. The cone is the lower bound for z and the paraboloid is the upper bound. The projection of the region onto the xy-plane is the circle of radius 1 centered at the origin.
Thus, we can describe the region as
Equation:

$$E = \left\{(r, \theta, z) | 0 \leq \theta \leq 2\pi, 0 \leq r \leq 1, r \leq z \leq 2 - r^2\right\}.$$

Hence the integral for the volume is
Equation:

$$V = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \int_{z=r}^{z=2-r^2} r \, dz \, dr \, d\theta.$$

b. We can also write the cone surface as $r = z$ and the paraboloid as $r^2 = 2 - z$. The lower bound for r is zero, but the upper bound is sometimes the cone and the other times it is the paraboloid. The plane $z = 1$ divides the region into two regions. Then the region can be described as
Equation:

$$\begin{aligned} E &= \{(r, \theta, z) | 0 \leq \theta \leq 2\pi, 0 \leq z \leq 1, 0 \leq r \leq z\} \\ &\cup \left\{(r, \theta, z) | 0 \leq \theta \leq 2\pi, 1 \leq z \leq 2, 0 \leq r \leq \sqrt{2 - z}\right\}. \end{aligned}$$

Now the integral for the volume becomes
Equation:

$$V = \int_{\theta=0}^{\theta=2\pi} \int_{z=0}^{z=1} \int_{r=0}^{r=z} r\,dr\,dz\,d\theta + \int_{\theta=0}^{\theta=2\pi} \int_{z=1}^{z=2} \int_{r=0}^{r=\sqrt{2-z}} r\,dr\,dz\,d\theta.$$

Note:
Exercise:

Problem: Redo the previous example with the order of integration $d\theta\,dz\,dr$.

Solution:

$E = \left\{(r,\theta,z) | 0 \le \theta \le 2\pi, 0 \le z \le 1, z \le r \le 2 - z^2\right\}$ and $V = \int_{r=0}^{r=1} \int_{z=r}^{z=2-r^2} \int_{\theta=0}^{\theta=2\pi} r\,d\theta\,dz\,dr.$

Hint

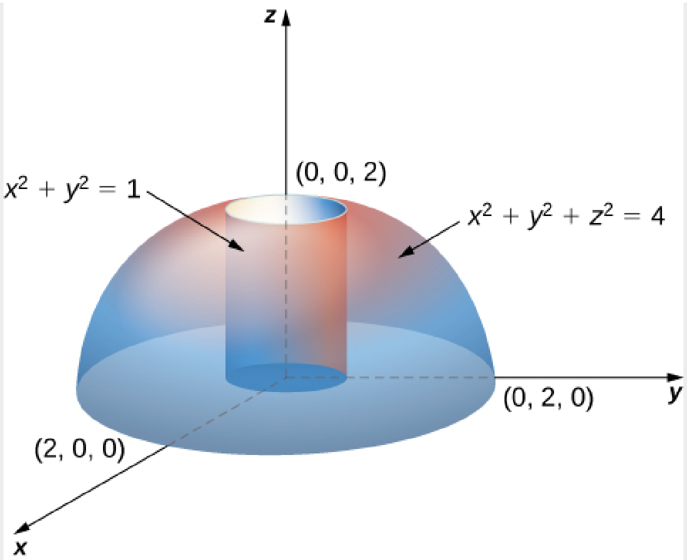
Note that θ is independent of r and z.

Example:
Exercise:

Problem:
Finding a Volume with Triple Integrals in Two Ways

Let E be the region bounded below by the $r\theta$-plane, above by the sphere $x^2 + y^2 + z^2 = 4$, and on the sides by the cylinder $x^2 + y^2 = 1$ ([link]). Set up a triple integral in cylindrical coordinates to find the volume of the region using the following orders of integration, and in each case find the volume and check that the answers are the same:

a. $dz\,dr\,d\theta$
b. $dr\,dz\,d\theta$.



Finding a cylindrical volume with a triple integral in cylindrical coordinates.

Solution:

a. Note that the equation for the sphere is

Equation:

$$x^2 + y^2 + z^2 = 4 \text{ or } r^2 + z^2 = 4$$

and the equation for the cylinder is

Equation:

$$x^2 + y^2 = 1 \text{ or } r^2 = 1.$$

Thus, we have for the region E

Equation:

$$E = \left\{(r, \theta, z) | 0 \leq z \leq \sqrt{4 - r^2}, 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\right\}$$

Hence the integral for the volume is

Equation:

$$\begin{aligned} V(E) &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \int_{z=0}^{z=\sqrt{4-r^2}} r\, dz\, dr\, d\theta \\ &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \left[rz|_{z=0}^{z=\sqrt{4-r^2}} \right] dr\, d\theta = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \left(r\sqrt{4-r^2} \right) dr\, d\theta \\ &= \int_0^{2\pi} \left(\frac{8}{3} - \sqrt{3} \right) d\theta = 2\pi \left(\frac{8}{3} - \sqrt{3} \right) \text{ cubic units.} \end{aligned}$$

b. Since the sphere is $x^2 + y^2 + z^2 = 4$, which is $r^2 + z^2 = 4$, and the cylinder is $x^2 + y^2 = 1$, which is $r^2 = 1$, we have $1 + z^2 = 4$, that is, $z^2 = 3$. Thus we have two regions, since the sphere and the cylinder intersect at $\left(1, \sqrt{3}\right)$ in the rz-plane

Equation:

$$E_1 = \left\{ (r, \theta, z) | 0 \leq r \leq \sqrt{4-r^2}, \sqrt{3} \leq z \leq 2, 0 \leq \theta \leq 2\pi \right\}$$

and

Equation:

$$E_2 = \left\{ (r, \theta, z) | 0 \leq r \leq 1, 0 \leq z \leq \sqrt{3}, 0 \leq \theta \leq 2\pi \right\}.$$

Hence the integral for the volume is

Equation:

$$\begin{aligned} V(E) &= \int_{\theta=0}^{\theta=2\pi} \int_{z=\sqrt{3}}^{z=2} \int_{r=0}^{r=\sqrt{4-r^2}} r\, dr\, dz\, d\theta + \int_{\theta=0}^{\theta=2\pi} \int_{z=0}^{z=\sqrt{3}} \int_{r=0}^{r=1} r\, dr\, dz\, d\theta \\ &= \sqrt{3}\pi + \left(\tfrac{16}{3} - 3\sqrt{3} \right) \pi = 2\pi \left(\tfrac{8}{3} - \sqrt{3} \right) \text{ cubic units.} \end{aligned}$$

Note:
Exercise:

Problem: Redo the previous example with the order of integration $d\theta\, dz\, dr$.

Solution:

$E_2 = \left\{(r, \theta, z) | 0 \le \theta \le 2\pi, 0 \le r \le 1, r \le z \le \sqrt{4 - r^2}\right\}$ and

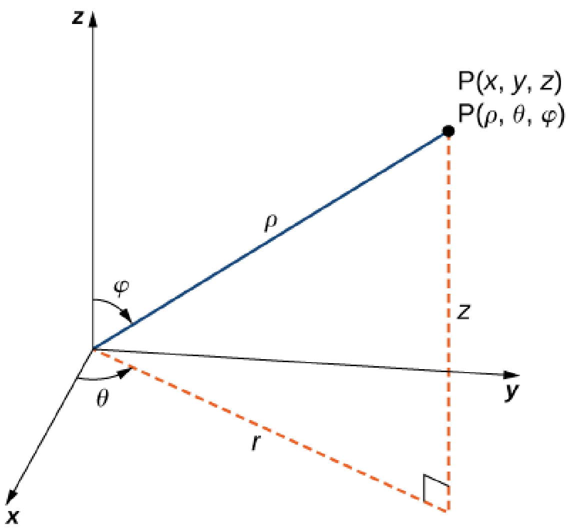
$$V = \int_{r=0}^{r=1} \int_{z=r}^{z=\sqrt{4-r^2}} \int_{\theta=0}^{\theta=2\pi} r \, d\theta \, dz \, dr.$$

Hint

A figure can be helpful. Note that θ is independent of r and z.

Review of Spherical Coordinates

In three-dimensional space $\mathbb{R}^3$ in the spherical coordinate system, we specify a point P by its distance ρ from the origin, the polar angle θ from the positive x-axis (same as in the cylindrical coordinate system), and the angle φ from the positive z-axis and the line OP ([link]). Note that $\rho \ge 0$ and $0 \le \varphi \le \pi$. (Refer to Cylindrical and Spherical Coordinates for a review.) Spherical coordinates are useful for triple integrals over regions that are symmetric with respect to the origin.



The spherical coordinate system locates points with two angles and a distance from the origin.

Recall the relationships that connect rectangular coordinates with spherical coordinates.

From spherical coordinates to rectangular coordinates:
Equation:

$$x = \rho \sin\varphi \cos\theta, y = \rho \sin\varphi \sin\theta, \text{ and } z = \rho \cos\varphi.$$

From rectangular coordinates to spherical coordinates:
Equation:

$$\rho^2 = x^2 + y^2 + z^2, \tan\theta = \frac{y}{x}, \varphi = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right).$$

Other relationships that are important to know for conversions are
Equation:

- $r = \rho \sin\varphi$
- $\theta = \theta$
- $z = \rho \cos\varphi$

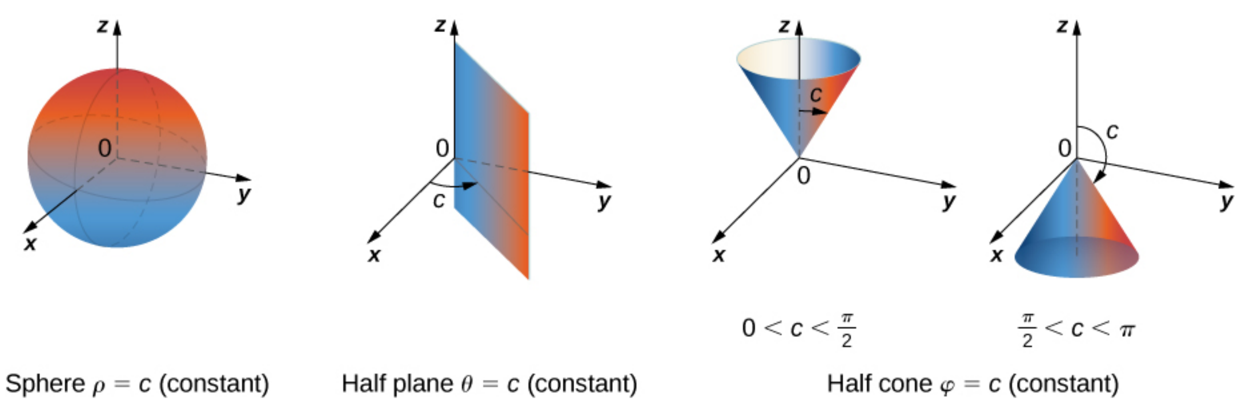
These equations are used to convert from spherical coordinates to cylindrical coordinates

and
Equation:

- $\rho = \sqrt{r^2 + z^2}$
- $\theta = \theta$
- $\varphi = \arccos\left(\frac{z}{\sqrt{r^2+z^2}}\right)$

These equations are used to convert from cylindrical coordinates to spherical coordinates.

The following figure shows a few solid regions that are convenient to express in spherical coordinates.

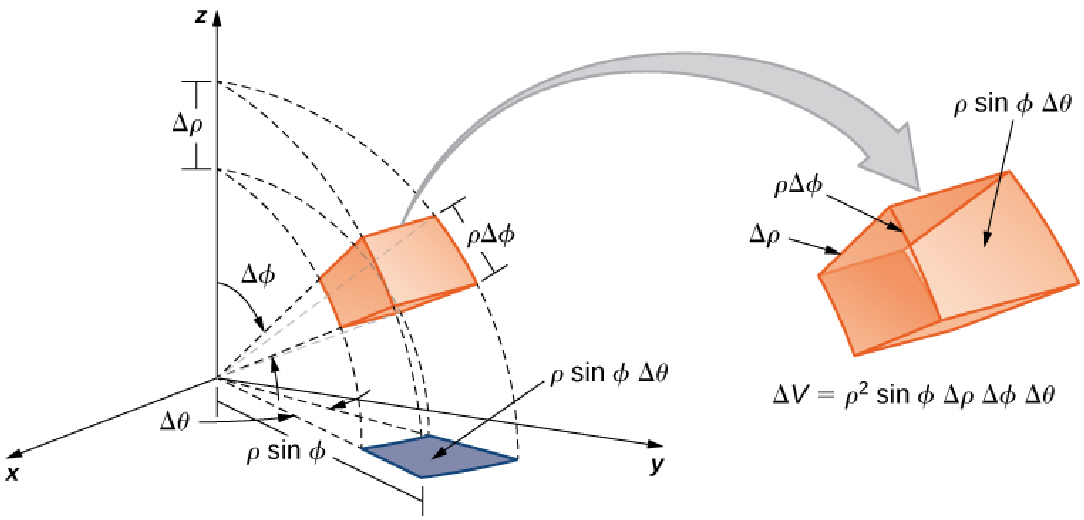


Spherical coordinates are especially convenient for working with solids bounded by these types of surfaces. (The letter c indicates a constant.)

Integration in Spherical Coordinates

We now establish a triple integral in the spherical coordinate system, as we did before in the cylindrical coordinate system. Let the function $f(\rho,\theta,\varphi)$ be continuous in a bounded spherical box, $B=\{(\rho,\theta,\varphi)|a\le\rho\le b,\alpha\le\theta\le\beta,\gamma\le\varphi\le\psi\}$. We then divide each interval into l,m and n subdivisions such that $\Delta\rho=\frac{b-a}{l},\Delta\theta=\frac{\beta-\alpha}{m},\Delta\varphi=\frac{\psi-\gamma}{n}$.

Now we can illustrate the following theorem for triple integrals in spherical coordinates with $(\overset{*}{\rho}_{ijk},\overset{*}{\theta}_{ijk},\overset{*}{\varphi}_{ijk})$ being any sample point in the spherical subbox B_{ijk}. For the volume element of the subbox ΔV in spherical coordinates, we have $\Delta V=(\Delta\rho)(\rho\Delta\varphi)(\rho\sin\varphi\Delta\theta),$, as shown in the following figure.



The volume element of a box in spherical coordinates.

Note:
Definition
The **triple integral in spherical coordinates** is the limit of a triple Riemann sum,
Equation:

$$\lim_{l,m,n\to\infty}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n}f(\overset{*}{\rho}_{ijk},\overset{*}{\theta}_{ijk},\overset{*}{\varphi}_{ijk})(\overset{*}{\rho}_{ijk})^2\sin\varphi\Delta\rho\Delta\theta\Delta\varphi$$

provided the limit exists.

As with the other multiple integrals we have examined, all the properties work similarly for a triple integral in the spherical coordinate system, and so do the iterated integrals. Fubini’s theorem takes the following form.

Note:
Fubini's Theorem for Spherical Coordinates
If $f(\rho, \theta, \varphi)$ is continuous on a spherical solid box $B = [a, b] \times [\alpha, \beta] \times [\gamma, \psi]$, then
Equation:

$$\iiint_B f(\rho, \theta, \varphi) \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = \int_{\varphi=\gamma}^{\varphi=\psi} \int_{\theta=\alpha}^{\theta=\beta} \int_{\rho=a}^{\rho=b} f(\rho, \theta, \varphi) \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta.$$

This iterated integral may be replaced by other iterated integrals by integrating with respect to the three variables in other orders.

As stated before, spherical coordinate systems work well for solids that are symmetric around a point, such as spheres and cones. Let us look at some examples before we consider triple integrals in spherical coordinates on general spherical regions.

Example:
Exercise:

Problem:
Evaluating a Triple Integral in Spherical Coordinates

Evaluate the iterated triple integral $\displaystyle\int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\pi/2} \int_{p=0}^{\rho=1} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta.$

Solution:

As before, in this case the variables in the iterated integral are actually independent of each other and hence we can integrate each piece and multiply:
Equation:

$$\int_0^{2\pi} \int_0^{\pi/2} \int_0^1 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = \int_0^{2\pi} d\theta \int_0^{\pi/2} \sin\varphi \, d\varphi \int_0^1 \rho^2 d\rho = (2\pi)(1)\left(\frac{1}{3}\right) = \frac{2\pi}{3}.$$

The concept of triple integration in spherical coordinates can be extended to integration over a general solid, using the projections onto the coordinate planes. Note that dV and dA mean the increments in volume and area, respectively. The variables V and A are used as the variables for integration to express the integrals.

The triple integral of a continuous function $f(\rho, \theta, \varphi)$ over a general solid region
Equation:

$$E = \{(\rho, \theta, \varphi) | (\rho, \theta) \in D, u_1(\rho, \theta) \le \varphi \le u_2(\rho, \theta)\}$$

in $\mathbb{R}^3$, where D is the projection of E onto the $\rho\theta$-plane, is
Equation:

$$\iiint\limits_E f(\rho,\theta,\varphi)dV = \iint\limits_D \left[\int\limits_{u_1(\rho,\theta)}^{u_2(\rho,\theta)} f(\rho,\theta,\varphi)d\varphi\right]dA.$$

In particular, if $D = \{(\rho,\theta)|g_1(\theta) \le \rho \le g_2(\theta), \alpha \le \theta \le \beta\}$, then we have
Equation:

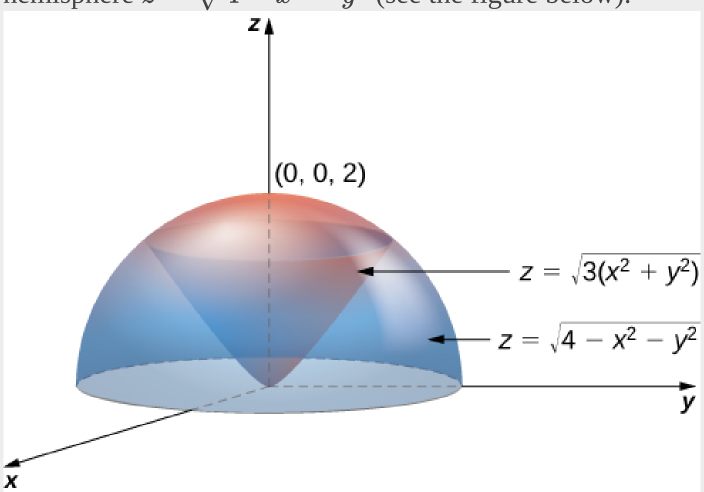
$$\iiint\limits_E f(\rho,\theta,\varphi)dV = \int\limits_{\alpha}^{\beta}\int\limits_{g_1(\theta)}^{g_2(\theta)}\int\limits_{u_1(\rho,\theta)}^{u_2(\rho,\theta)} f(\rho,\theta,\varphi)\rho^2 \sin\varphi\, d\varphi\, d\rho\, d\theta.$$

Similar formulas occur for projections onto the other coordinate planes.

Example:
Exercise:

Problem:
Setting up a Triple Integral in Spherical Coordinates

Set up an integral for the volume of the region bounded by the cone $z = \sqrt{3(x^2+y^2)}$ and the hemisphere $z = \sqrt{4-x^2-y^2}$ (see the figure below).



A region bounded below by a cone and above by a hemisphere.

Solution:

Using the conversion formulas from rectangular coordinates to spherical coordinates, we have:

For the cone: $z = \sqrt{3\left(x^2 + y^2\right)}$ or $\rho \cos \varphi = \sqrt{3}\rho \sin \varphi$ or $\tan \varphi = \frac{1}{\sqrt{3}}$ or $\varphi = \frac{\pi}{6}$.

For the sphere: $z = \sqrt{4 - x^2 - y^2}$ or $z^2 + x^2 + y^2 = 4$ or $\rho^2 = 4$ or $\rho = 2$.

Thus, the triple integral for the volume is $V\left(E\right) = \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\varphi=\pi/6} \int_{\rho=0}^{\rho=2} \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta.$

Note:
Exercise:

Problem:

Set up a triple integral for the volume of the solid region bounded above by the sphere $\rho = 2$ and bounded below by the cone $\varphi = \pi/3$.

Solution:

$$V\left(E\right) = \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\varphi=\pi/3} \int_{\rho=0}^{\rho=2} \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta$$

Hint

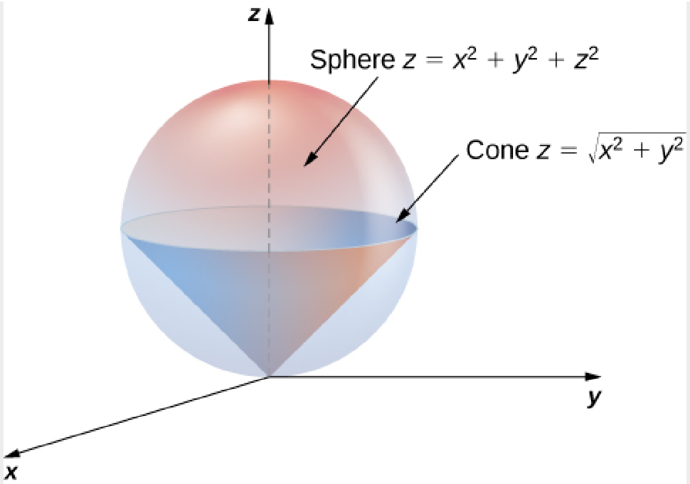
Follow the steps of the previous example.

Example:
Exercise:

Problem:
Interchanging Order of Integration in Spherical Coordinates

Let E be the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the sphere $z = x^2 + y^2 + z^2$ ([link]). Set up a triple integral in spherical coordinates and find the volume of the region using the following orders of integration:

a. $d\rho \, d\phi \, d\theta$,
b. $d\varphi \, d\rho \, d\theta$.



A region bounded below by a cone and above by a sphere.

Solution:

a. Use the conversion formulas to write the equations of the sphere and cone in spherical coordinates.
For the sphere:
Equation:

$$\begin{aligned} x^2 + y^2 + z^2 &= z \\ \rho^2 &= \rho \cos \varphi \\ \rho &= \cos \varphi. \end{aligned}$$

For the cone:
Equation:

$$\begin{aligned} z &= \sqrt{x^2 + y^2} \\ \rho \cos \varphi &= \sqrt{\rho^2 \sin^2 \varphi \cos^2 \phi + \rho^2 \sin^2 \varphi \sin^2 \phi} \\ \rho \cos \varphi &= \sqrt{\rho^2 \sin^2 \varphi \left(\cos^2 \phi + \sin^2 \phi \right)} \\ \rho \cos \varphi &= \rho \sin \varphi \\ \cos \varphi &= \sin \varphi \\ \varphi &= \pi/4. \end{aligned}$$

Hence the integral for the volume of the solid region E becomes
Equation:

$$V(E) = \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\pi/4} \int_{\rho=0}^{\rho=\cos\varphi} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta.$$

b. Consider the $\varphi\rho$-plane. Note that the ranges for φ and ρ (from part a.) are
Equation:

$$\begin{aligned} 0 \le \varphi \le \pi/4 \\ 0 \le \rho \le \cos\varphi. \end{aligned}$$

The curve $\rho = \cos\varphi$ meets the line $\varphi = \pi/4$ at the point $\left(\pi/4, \sqrt{2}/2\right)$. Thus, to change the order of integration, we need to use two pieces:
Equation:

$$\begin{aligned} 0 \le \rho \le \sqrt{2}/2 \\ 0 \le \varphi \le \pi/4 \end{aligned} \quad \text{and} \quad \begin{aligned} \sqrt{2}/2 \ \le \ \rho \le 1 \\ 0 \ \le \ \varphi \le \cos^{-1}\rho. \end{aligned}$$

Hence the integral for the volume of the solid region E becomes
Equation:

$$V(E) = \int_{\theta=0}^{\theta=2\pi} \int_{\rho=0}^{\rho=\sqrt{2}/2} \int_{\varphi=0}^{\varphi=\pi/4} \rho^2 \sin\varphi \, d\varphi \, d\rho \, d\theta + \int_{\theta=0}^{\theta=2\pi} \int_{\rho=\sqrt{2}/2}^{\rho=1} \int_{\varphi=0}^{\varphi=\cos^{-1}\rho} \rho^2 \sin\varphi \, d\varphi \, d\rho \, d\theta.$$

In each case, the integration results in $V(E) = \frac{\pi}{8}$.

Before we end this section, we present a couple of examples that can illustrate the conversion from rectangular coordinates to cylindrical coordinates and from rectangular coordinates to spherical coordinates.

Example:
Exercise:

Problem:
Converting from Rectangular Coordinates to Cylindrical Coordinates

Convert the following integral into cylindrical coordinates:
Equation:

$$\int_{y=-1}^{y=1} \int_{x=0}^{x=\sqrt{1-y^2}} \int_{z=x^2+y^2}^{z=\sqrt{x^2+y^2}} xyz \, dz \, dx \, dy.$$

Solution:

The ranges of the variables are

Equation:

$$\begin{aligned} -1 &\leq y \leq 1 \\ 0 &\leq x \leq \sqrt{1-y^2} \\ x^2+y^2 &\leq z \leq \sqrt{x^2+y^2}. \end{aligned}$$

The first two inequalities describe the right half of a circle of radius 1. Therefore, the ranges for θ and r are

Equation:

$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \text{ and } 0 \leq r \leq 1.$$

The limits of z are $r^2 \leq z \leq r$, hence

Equation:

$$\int_{y=-1}^{y=1} \int_{x=0}^{x=\sqrt{1-y^2}} \int_{z=x^2+y^2}^{z=\sqrt{x^2+y^2}} xyz\,dz\,dx\,dy = \int_{\theta=-\pi/2}^{\theta=\pi/2} \int_{r=0}^{r=1} \int_{z=r^2}^{z=r} r\,(r\cos\theta)\,(r\sin\theta) z\,dz\,dr\,d\theta.$$

Example:

Exercise:

Problem:

Converting from Rectangular Coordinates to Spherical Coordinates

Convert the following integral into spherical coordinates:

Equation:

$$\int_{y=0}^{y=3} \int_{x=0}^{x=\sqrt{9-y^2}} \int_{z=\sqrt{x^2+y^2}}^{z=\sqrt{18-x^2-y^2}} \left(x^2+y^2+z^2\right) dz\,dx\,dy.$$

Solution:

The ranges of the variables are

Equation:

$$\begin{aligned} 0 &\le y \le 3 \\ 0 &\le x \le \sqrt{9-y^2} \\ \sqrt{x^2+y^2} &\le z \le \sqrt{18-x^2-y^2}. \end{aligned}$$

The first two ranges of variables describe a quarter disk in the first quadrant of the xy-plane. Hence the range for θ is $0 \le \theta \le \frac{\pi}{2}$.

The lower bound $z = \sqrt{x^2+y^2}$ is the upper half of a cone and the upper bound $z = \sqrt{18-x^2-y^2}$ is the upper half of a sphere. Therefore, we have $0 \le \rho \le \sqrt{18}$, which is $0 \le \rho \le 3\sqrt{2}$.

For the ranges of φ, we need to find where the cone and the sphere intersect, so solve the equation
Equation:

$$\begin{aligned} r^2+z^2 &= 18 \\ \left(\sqrt{x^2+y^2}\right)^2+z^2 &= 18 \\ z^2+z^2 &= 18 \\ 2z^2 &= 18 \\ z^2 &= 9 \\ z &= 3. \end{aligned}$$

This gives
Equation:

$$\begin{aligned} 3\sqrt{2}\cos\varphi &= 3 \\ \cos\varphi &= \frac{1}{\sqrt{2}} \\ \varphi &= \frac{\pi}{4}. \end{aligned}$$

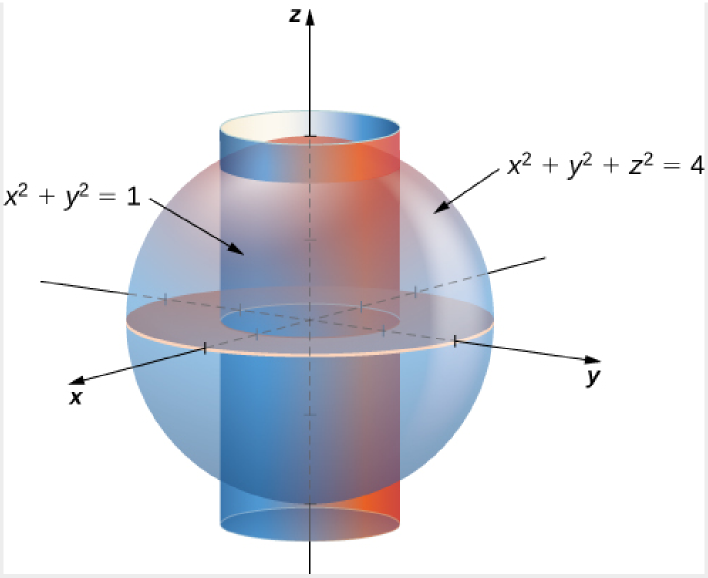
Putting this together, we obtain
Equation:

$$\int_{y=0}^{y=3} \int_{x=0}^{x=\sqrt{9-y^2}} \int_{z=\sqrt{x^2+y^2}}^{z=\sqrt{18-x^2-y^2}} \left(x^2+y^2+z^2\right) dz\,dx\,dy = \int_{\varphi=0}^{\varphi=\pi/4} \int_{\theta=0}^{\theta=\pi/2} \int_{\rho=0}^{\rho=3\sqrt{2}} \rho^4 \sin\varphi\,d\rho\,d\theta\,d\varphi.$$

Note:
Exercise:

Problem:

Use rectangular, cylindrical, and spherical coordinates to set up triple integrals for finding the volume of the region inside the sphere $x^2+y^2+z^2=4$ but outside the cylinder $x^2+y^2=1$.



Solution:

Rectangular: $\displaystyle\int_{x=-2}^{x=2}\int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}}\int_{z=-\sqrt{4-x^2-y^2}}^{z=\sqrt{4-x^2-y^2}} dz\,dy\,dx - \int_{x=-1}^{x=1}\int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}}\int_{z=-\sqrt{4-x^2-y^2}}^{z=\sqrt{4-x^2-y^2}} dz\,dy\,dx.$

Cylindrical: $\displaystyle\int_{\theta=0}^{\theta=2\pi}\int_{r=1}^{r=2}\int_{z=-\sqrt{4-r^2}}^{z=\sqrt{4-r^2}} r\,dz\,dr\,d\theta.$

Spherical: $\displaystyle\int_{\varphi=\pi/6}^{\varphi=5\pi/6}\int_{\theta=0}^{\theta=2\pi}\int_{\rho=\csc\varphi}^{\rho=2} \rho^2 \sin\varphi\,d\rho\,d\theta\,d\varphi.$

Now that we are familiar with the spherical coordinate system, let’s find the volume of some known geometric figures, such as spheres and ellipsoids.

Example:
Exercise:

Problem:
Chapter Opener: Finding the Volume of l’Hemisphèric

Find the volume of the spherical planetarium in l’Hemisphèric in Valencia, Spain, which is five stories tall and has a radius of approximately 50 ft, using the equation $x^2 + y^2 + z^2 = r^2$.

(credit: modification of work by Javier Yaya Tur, Wikimedia Commons)

Solution:

We calculate the volume of the ball in the first octant, where $x \geq 0, y \geq 0$, and $z \geq 0$, using spherical coordinates, and then multiply the result by 8 for symmetry. Since we consider the region D as the first octant in the integral, the ranges of the variables are

Equation:

$$0 \leq \varphi \leq \frac{\pi}{2}, 0 \leq \rho \leq r, 0 \leq \theta \leq \frac{\pi}{2}.$$

Therefore,

Equation:

$$\begin{aligned} V &= \iiint_D dx\,dy\,dz = 8 \int_{\theta=0}^{\theta=\pi/2} \int_{\rho=0}^{\rho=\pi} \int_{\varphi=0}^{\varphi=\pi/2} \rho^2 \sin\theta\,d\varphi\,d\rho\,d\theta \\ &= 8 \int_{\varphi=0}^{\varphi=\pi/2} d\varphi \int_{\rho=0}^{\rho=r} \rho^2 d\rho \int_{\theta=0}^{\theta=\pi/2} \sin\theta\,d\theta \\ &= 8\left(\frac{\pi}{2}\right)\left(\frac{r^3}{3}\right)(1) \\ &= \frac{4}{3}\pi r^3. \end{aligned}$$

This exactly matches with what we knew. So for a sphere with a radius of approximately 50 ft, the volume is $\frac{4}{3}\pi(50)^3 \approx 523{,}600\,\text{ft}^3$.

For the next example we find the volume of an ellipsoid.

Example:
Exercise:

Problem:
Finding the Volume of an Ellipsoid

Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$

Solution:

We again use symmetry and evaluate the volume of the ellipsoid using spherical coordinates. As before, we use the first octant $x \geq 0, y \geq 0$, and $z \geq 0$ and then multiply the result by 8.

In this case the ranges of the variables are
Equation:

$$0 \leq \varphi \leq \frac{\pi}{2}, 0 \leq \rho \leq \frac{\pi}{2}, 0 \leq \rho \leq 1, \text{ and } 0 \leq \theta \leq \frac{\pi}{2}.$$

Also, we need to change the rectangular to spherical coordinates in this way:
Equation:

$$x = a\rho \cos \varphi \sin \theta, y = b\rho \sin \varphi \sin \theta, \text{ and } z = c\rho \cos \theta.$$

Then the volume of the ellipsoid becomes
Equation:

$$\begin{aligned} V &= \iiint\limits_D dx\, dy\, dz \\ &= 8 \int\limits_{\theta=0}^{\theta=\pi/2} \int\limits_{\rho=0}^{\rho=1} \int\limits_{\varphi=0}^{\varphi=\pi/2} abc\rho^2 \sin \theta \, d\varphi \, d\rho \, d\theta \\ &= 8abc \int\limits_{\varphi=0}^{\varphi=\pi/2} d\varphi \int\limits_{\rho=0}^{\rho=1} \rho^2 d\rho \int\limits_{\theta=0}^{\theta=\pi/2} \sin \theta \, d\theta \\ &= 8abc \left(\frac{\pi}{2}\right) \left(\frac{1}{3}\right) (1) \\ &= \frac{4}{3} \pi abc. \end{aligned}$$

Example:
Exercise:

Problem:
Finding the Volume of the Space Inside an Ellipsoid and Outside a Sphere

Find the volume of the space inside the ellipsoid $\frac{x^2}{75^2} + \frac{y^2}{80^2} + \frac{z^2}{90^2} = 1$ and outside the sphere $x^2 + y^2 + z^2 = 50^2$.

Solution:

This problem is directly related to the l'Hemisphèric structure. The volume of space inside the ellipsoid and outside the sphere might be useful to find the expense of heating or cooling that space. We can use the preceding two examples for the volume of the sphere and ellipsoid and then substract.

First we find the volume of the ellipsoid using $a = 75\,\text{ft}$, $b = 80\,\text{ft}$, and $c = 90\,\text{ft}$ in the result from [link]. Hence the volume of the ellipsoid is

Equation:

$$V_{\text{ellipsoid}} = \frac{4}{3}\pi(75)(80)(90) \approx 2{,}262{,}000\,\text{ft}^3.$$

From [link], the volume of the sphere is

Equation:

$$V_{\text{sphere}} \approx 523{,}600\,\text{ft}^3.$$

Therefore, the volume of the space inside the ellipsoid $\frac{x^2}{75^2} + \frac{y^2}{80^2} + \frac{z^2}{90^2} = 1$ and outside the sphere $x^2 + y^2 + z^2 = 50^2$ is approximately

Equation:

$$V_{\text{Hemisferic}} = V_{\text{ellipsoid}} - V_{\text{sphere}} = 1{,}738{,}400\,\text{ft}^3.$$

Note:

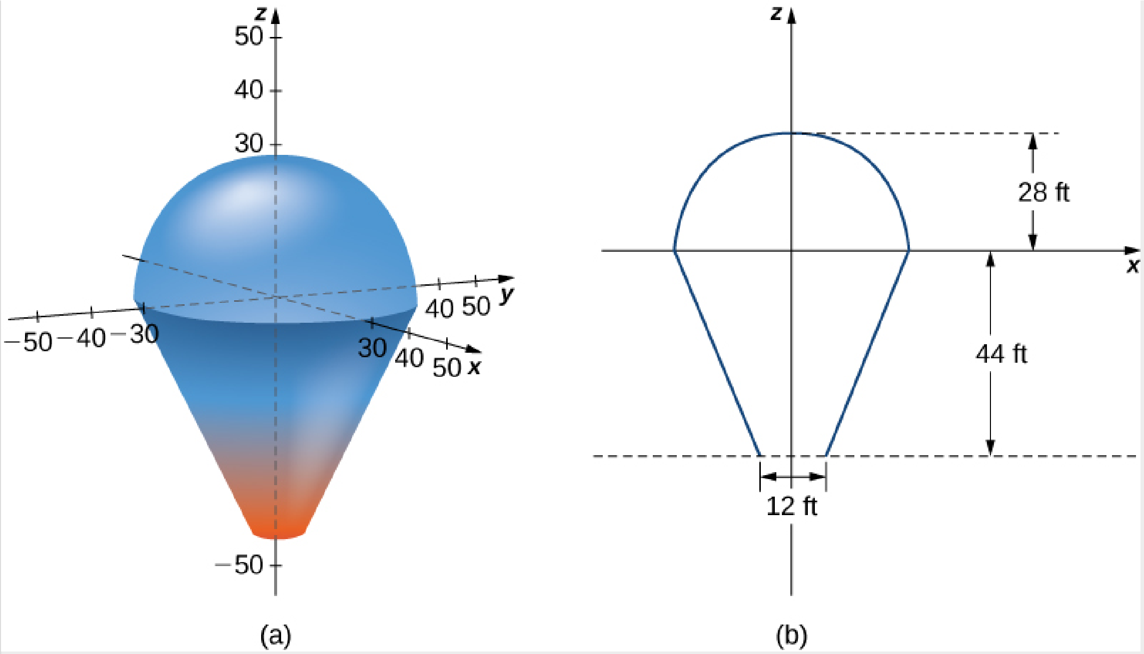
Hot air balloons

Hot air ballooning is a relaxing, peaceful pastime that many people enjoy. Many balloonist gatherings take place around the world, such as the Albuquerque International Balloon Fiesta. The Albuquerque event is the largest hot air balloon festival in the world, with over 500 balloons participating each year.

Balloons lift off at the 2001 Albuquerque International Balloon Fiesta. (credit: David Herrera, Flickr)

As the name implies, hot air balloons use hot air to generate lift. (Hot air is less dense than cooler air, so the balloon floats as long as the hot air stays hot.) The heat is generated by a propane burner suspended below the opening of the basket. Once the balloon takes off, the pilot controls the altitude of the balloon, either by using the burner to heat the air and ascend or by using a vent near the top of the balloon to release heated air and descend. The pilot has very little control over where the balloon goes, however—balloons are at the mercy of the winds. The uncertainty over where we will end up is one of the reasons balloonists are attracted to the sport.

In this project we use triple integrals to learn more about hot air balloons. We model the balloon in two pieces. The top of the balloon is modeled by a half sphere of radius 28 feet. The bottom of the balloon is modeled by a frustum of a cone (think of an ice cream cone with the pointy end cut off). The radius of the large end of the frustum is 28 feet and the radius of the small end of the frustum is 6 feet. A graph of our balloon model and a cross-sectional diagram showing the dimensions are shown in the following figure.



(a) Use a half sphere to model the top part of the balloon and a frustum of a cone to model the bottom part of the balloon. (b) A cross section of the balloon showing its dimensions.

We first want to find the volume of the balloon. If we look at the top part and the bottom part of the balloon separately, we see that they are geometric solids with known volume formulas. However, it is still worthwhile to set up and evaluate the integrals we would need to find the volume. If we calculate the volume using integration, we can use the known volume formulas to check our answers. This will help ensure that we have the integrals set up correctly for the later, more complicated stages of the project.

1. Find the volume of the balloon in two ways.

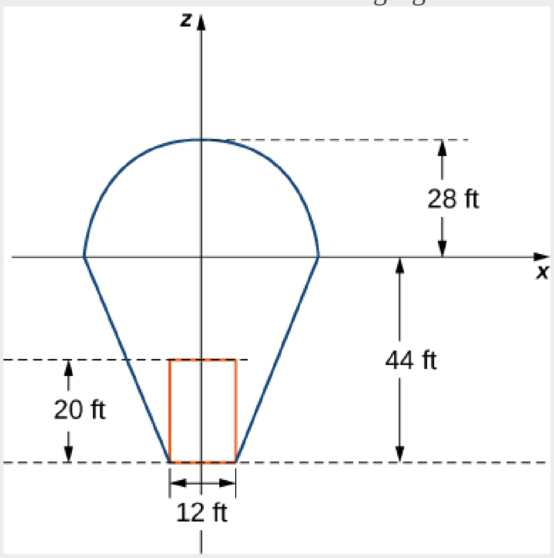
   a. Use triple integrals to calculate the volume. Consider each part of the balloon separately. (Consider using spherical coordinates for the top part and cylindrical coordinates for the bottom part.)
   b. Verify the answer using the formulas for the volume of a sphere, $V = \frac{4}{3}\pi r^3$, and for the volume of a cone, $V = \frac{1}{3}\pi r^2 h$.

   In reality, calculating the temperature at a point inside the balloon is a tremendously complicated endeavor. In fact, an entire branch of physics (thermodynamics) is devoted to studying heat and temperature. For the purposes of this project, however, we are going to make some simplifying assumptions about how temperature varies from point to point within the balloon. Assume that just prior to liftoff, the temperature (in degrees Fahrenheit) of the air inside the balloon varies according to the function
   Equation:

$$T_0(r, \theta, z) = \frac{z - r}{10} + 210.$$

2. What is the average temperature of the air in the balloon just prior to liftoff? (Again, look at each part of the balloon separately, and do not forget to convert the function into spherical coordinates when looking at the top part of the balloon.)

Now the pilot activates the burner for 10 seconds. This action affects the temperature in a 12-foot-wide column 20 feet high, directly above the burner. A cross section of the balloon depicting this column in shown in the following figure.



Activating the burner heats the air in a 20-foot-high, 12-foot-wide column directly above the burner.

Assume that after the pilot activates the burner for 10 seconds, the temperature of the air in the column described above *increases* according to the formula

Equation:

$$H(r, \theta, z) = -2z - 48.$$

Then the temperature of the air in the column is given by

Equation:

$$T_1(r, \theta, z) = \frac{z - r}{10} + 210 + (-2z - 48),$$

while the temperature in the remainder of the balloon is still given by

Equation:

$$T_0(r, \theta, z) = \frac{z - r}{10} + 210.$$

3. Find the average temperature of the air in the balloon after the pilot has activated the burner for 10 seconds.

Key Concepts

- To evaluate a triple integral in cylindrical coordinates, use the iterated integral
Equation:

$$\int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} \int_{z=u_1(r,\theta)}^{z=u_2(r,\theta)} f(r,\theta,z) r \, dz \, dr \, d\theta.$$

- To evaluate a triple integral in spherical coordinates, use the iterated integral
Equation:

$$\int_{\theta=\alpha}^{\theta=\beta} \int_{\rho=g_1(\theta)}^{\rho=g_2(\theta)} \int_{\varphi=u_1(r,\theta)}^{\varphi=u_2(r,\theta)} f(\rho,\theta,\varphi) \rho^2 \sin\varphi \, d\varphi \, d\rho \, d\theta.$$

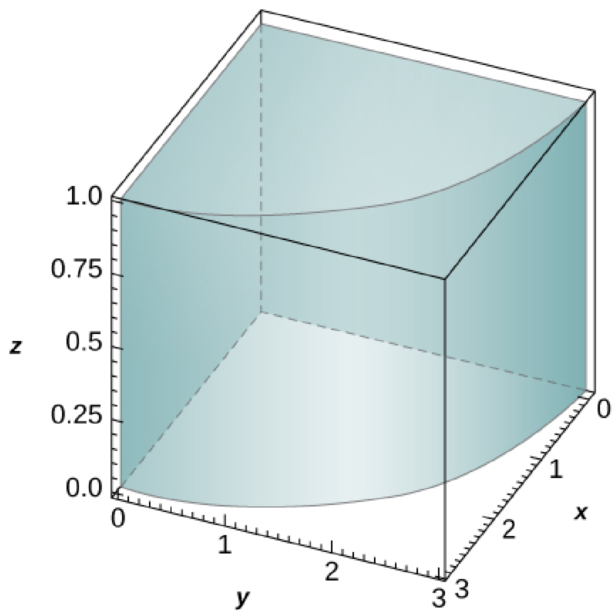
Key Equations

- **Triple integral in cylindrical coordinates**
$$\iiint_B g(x,y,z) dV = \iiint_B g(r\cos\theta, r\sin\theta, z) r \, dr \, d\theta \, dz = \iiint_B f(r,\theta,z) r \, dr \, d\theta \, dz$$
- **Triple integral in spherical coordinates**
$$\iiint_B f(\rho,\theta,\varphi) \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = \int_{\varphi=\gamma}^{\varphi=\psi} \int_{\theta=\alpha}^{\theta=\beta} \int_{\rho=a}^{\rho=b} f(\rho,\theta,\varphi) \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

In the following exercises, evaluate the triple integrals $\iiint_E f(x,y,z) dV$ over the solid E.

Exercise:

Problem: $f(x,y,z) = z$, $B = \left\{(x,y,z) \middle| x^2 + y^2 \le 9, x \ge 0, y \ge 0, 0 \le z \le 1\right\}$



Solution:

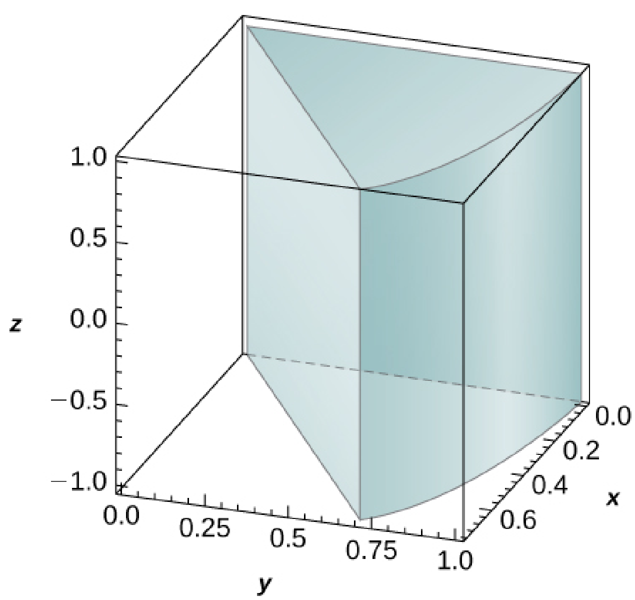
$\frac{9\pi}{8}$

Exercise:

Problem: $f(x, y, z) = xz^2$, $B = \left\{(x, y, z) \middle| x^2 + y^2 \leq 16, x \geq 0, y \leq 0, -1 \leq z \leq 1\right\}$

Exercise:

Problem: $f(x, y, z) = xy$, $B = \left\{(x, y, z) \middle| x^2 + y^2 \leq 1, x \geq 0, x \geq y, -1 \leq z \leq 1\right\}$



Solution:

$\frac{1}{8}$

Exercise:

Problem: $f(x,y,z)=x^2+y^2$, $B=\left\{(x,y,z)\middle|x^2+y^2\le 4, x\ge 0, x\le y, 0\le z\le 3\right\}$

Exercise:

Problem: $f(x,y,z)=e^{\sqrt{x^2+y^2}}$, $B=\left\{(x,y,z)\middle|1\le x^2+y^2\le 4, y\le 0, x\le y\sqrt{3}, 2\le z\le 3\right\}$

Solution:

$\frac{\pi e^2}{6}$

Exercise:

Problem: $f(x,y,z)=\sqrt{x^2+y^2}$, $B=\left\{(x,y,z)\middle|1\le x^2+y^2\le 9, y\le 0, 0\le z\le 1\right\}$

Exercise:

Problem:

a. Let B be a cylindrical shell with inner radius a, outer radius b, and height c, where $0<a<b$ and $c>0$. Assume that a function F defined on B can be expressed in cylindrical coordinates as $F(x,y,z)=f(r)+h(z)$, where f and h are differentiable functions. If $\int\limits_a^b \widetilde{f}(r)dr=0$ and $\widetilde{h}(0)=0$, where $\widetilde{f}$ and $\widetilde{h}$ are antiderivatives of f and h, respectively, show that
Equation:

$$\iiint\limits_B F\,(x,y,z)dV=2\pi c\left(b\widetilde{f}(b)-a\widetilde{f}(a)\right)+\pi\left(b^2-a^2\right)\widetilde{h}(c).$$

b. Use the previous result to show that $\iiint\limits_B\left(z+\sin\sqrt{x^2+y^2}\right)dx\,dy\,dz=6\pi^2\left(\pi-2\right)$, where B is a cylindrical shell with inner radius π, outer radius 2π, and height 2.

Exercise:

Problem:

a. Let B be a cylindrical shell with inner radius a, outer radius b, and height c, where $0<a<b$ and $c>0$. Assume that a function F defined on B can be expressed in cylindrical coordinates as $F(x,y,z)=f(r)g(\theta)h(z)$, where $f, g,$ and h are differentiable functions. If $\int\limits_a^b \widetilde{f}(r)dr=0$, where $\widetilde{f}$ is an antiderivative of f, show that
Equation:

$$\iiint_B F(x, y, z)dV = \left[b\widetilde{f}(b) - a\widetilde{f}(a)\right] \left[\tilde{g}(2\pi) - \tilde{g}(0)\right] \left[\widetilde{h}(c) - \widetilde{h}(0)\right],$$

where $\tilde{g}$ and $\widetilde{h}$ are antiderivatives of g and h, respectively.

b. Use the previous result to show that $\iiint_B z \sin\sqrt{x^2 + y^2} dx\, dy\, dz = -12\pi^2$, where B is a cylindrical shell with inner radius π, outer radius 2π, and height 2.

In the following exercises, the boundaries of the solid E are given in cylindrical coordinates.

a. Express the region E in cylindrical coordinates.
b. Convert the integral $\iiint_E f(x, y, z)dV$ to cylindrical coordinates.

Exercise:

Problem:

E is bounded by the right circular cylinder $r = 4 \sin\theta$, the $r\theta$-plane, and the sphere $r^2 + z^2 = 16$.

Solution:

a. $E = \left\{(r, \theta, z) \middle| 0 \leq \theta \leq \pi, 0 \leq r \leq 4\sin\theta, 0 \leq z \leq \sqrt{16 - r^2}\right\}$; b.
$$\int_0^{\pi} \int_0^{4\sin\theta} \int_0^{\sqrt{16-r^2}} f(r, \theta, z)r\, dz\, dr\, d\theta$$

Exercise:

Problem:

E is bounded by the right circular cylinder $r = \cos\theta$, the $r\theta$-plane, and the sphere $r^2 + z^2 = 9$.

Exercise:

Problem:

E is located in the first octant and is bounded by the circular paraboloid $z = 9 - 3r^2$, the cylinder $r = \sqrt{3}$, and the plane $r(\cos\theta + \sin\theta) = 20 - z$.

Solution:

a. $E = \left\{(r, \theta, z) \middle| 0 \leq \theta \leq \frac{\pi}{2}, 0 \leq r \leq \sqrt{3}, 9 - r^2 \leq z \leq 10 - r(\cos\theta + \sin\theta)\right\}$; b.
$$\int_0^{\pi/2} \int_0^{\sqrt{3}} \int_{9-r^2}^{10-r(\cos\theta+\sin\theta)} f(r, \theta, z)r\, dz\, dr\, d\theta$$

Exercise:

Problem:

E is located in the first octant outside the circular paraboloid $z=10-2r^2$ and inside the cylinder $r=\sqrt{5}$ and is bounded also by the planes $z=20$ and $\theta=\frac{\pi}{4}$.

In the following exercises, the function f and region E are given.

a. Express the region E and the function f in cylindrical coordinates.
b. Convert the integral $\iiint\limits_B f(x,y,z)dV$ into cylindrical coordinates and evaluate it.

Exercise:

Problem: $f(x,y,z)=\frac{1}{x+3}$, $E=\left\{(x,y,z)\middle|0\le x^2+y^2\le 9, x\ge 0, y\ge 0, 0\le z\le x+3\right\}$

Solution:

a. $E=\left\{(r,\theta,z)\middle|0\le r\le 3, 0\le\theta\le\frac{\pi}{2}, 0\le z\le r\cos\theta+3\right\}$, $f(r,\theta,z)=\frac{1}{r\cos\theta+3}$; b.
$$\int_0^3\int_0^{\pi/2}\int_0^{r\cos\theta+3}\frac{r}{r\cos\theta+3}dz\,d\theta\,dr=\frac{9\pi}{4}$$

Exercise:

Problem: $f(x,y,z)=x^2+y^2$, $E=\left\{(x,y,z)\middle|0\le x^2+y^2\le 4, y\ge 0, 0\le z\le 3-x\right\}$

Exercise:

Problem: $f(x,y,z)=x$, $E=\left\{(x,y,z)\middle|1\le y^2+z^2\le 9, 0\le x\le 1-y^2-z^2\right\}$

Solution:

a. $y=r\cos\theta$, $z=r\sin\theta$, $x=z$,
$E=\left\{(r,\theta,z)\middle|1\le r\le 3, 0\le\theta\le 2\pi, 0\le z\le 1-r^2\right\}$, $f(r,\theta,z)=z$; b.
$$\int_1^3\int_0^{2\pi}\int_0^{1-r^2}zr\,dz\,d\theta\,dr=\frac{356\pi}{3}$$

Exercise:

Problem: $f(x,y,z)=y$, $E=\left\{(x,y,z)\middle|1\le x^2+z^2\le 9, 0\le y\le 1-x^2-z^2\right\}$

In the following exercises, find the volume of the solid E whose boundaries are given in rectangular coordinates.

Exercise:

Problem: E is above the xy-plane, inside the cylinder $x^2+y^2=1$, and below the plane $z=1$.

Solution:

π

Exercise:

Problem: E is below the plane $z = 1$ and inside the paraboloid $z = x^2 + y^2$.

Exercise:

Problem: E is bounded by the circular cone $z = \sqrt{x^2 + y^2}$ and $z = 1$.

Solution:

$\frac{\pi}{3}$

Exercise:

Problem:

E is located above the xy-plane, below $z = 1$, outside the one-sheeted hyperboloid $x^2 + y^2 - z^2 = 1$, and inside the cylinder $x^2 + y^2 = 2$.

Exercise:

Problem:

E is located inside the cylinder $x^2 + y^2 = 1$ and between the circular paraboloids $z = 1 - x^2 - y^2$ and $z = x^2 + y^2$.

Solution:

π

Exercise:

Problem:

E is located inside the sphere $x^2 + y^2 + z^2 = 1$, above the xy-plane, and inside the circular cone $z = \sqrt{x^2 + y^2}$.

Exercise:

Problem:

E is located outside the circular cone $x^2 + y^2 = (z - 1)^2$ and between the planes $z = 0$ and $z = 2$.

Solution:

$\frac{4\pi}{3}$

Exercise:

Problem:

E is located outside the circular cone $z = 1 - \sqrt{x^2 + y^2}$, above the xy-plane, below the circular paraboloid, and between the planes $z = 0$ and $z = 2$.

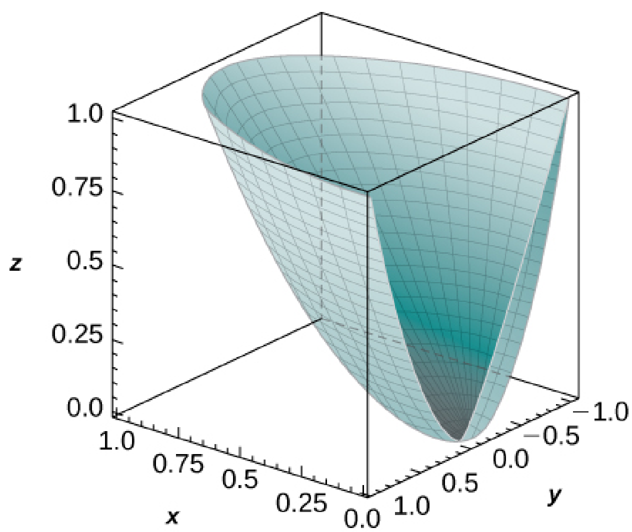
Exercise:

Problem:

[T] Use a computer algebra system (CAS) to graph the solid whose volume is given by the iterated integral in cylindrical coordinates $\int_{-\pi/2}^{\pi/2} \int_0^1 \int_{r^2}^{r} r\, dz\, dr\, d\theta$. Find the volume V of the solid. Round your answer to four decimal places.

Solution:

$V = \frac{\pi}{12} \approx 0.2618$



Exercise:

Problem:

[T] Use a CAS to graph the solid whose volume is given by the iterated integral in cylindrical coordinates $\int_0^{\pi/2} \int_0^1 \int_{r^4}^{r} r\, dz\, dr\, d\theta$. Find the volume V of the solid Round your answer to four decimal places.

Exercise:

Problem:

Convert the integral $\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{x^2+y^2}^{\sqrt{x^2+y^2}} xz\, dz\, dx\, dy$ into an integral in cylindrical coordinates.

Solution:

$$\int_0^1 \int_0^\pi \int_{r^2}^r zr^2 \cos\theta \, dz \, d\theta \, dr$$

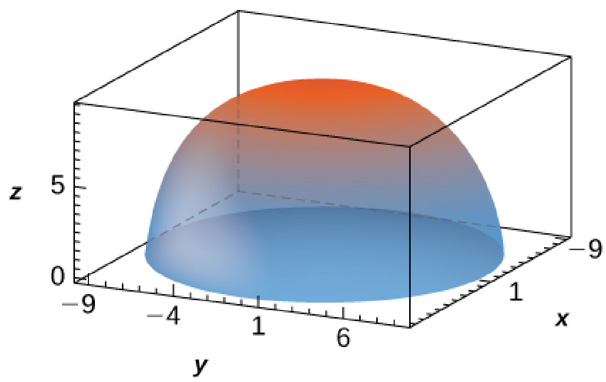
Exercise:

Problem: Convert the integral $\int_0^2 \int_0^x \int_0^1 (xy+z) dz \, dx \, dy$ into an integral in cylindrical coordinates.

In the following exercises, evaluate the triple integral $\iiint_B f(x,y,z) dV$ over the solid B.

Exercise:

Problem: $f(x,y,z)=1$, $B=\left\{(x,y,z) \middle| x^2+y^2+z^2 \le 90, z \ge 0\right\}$

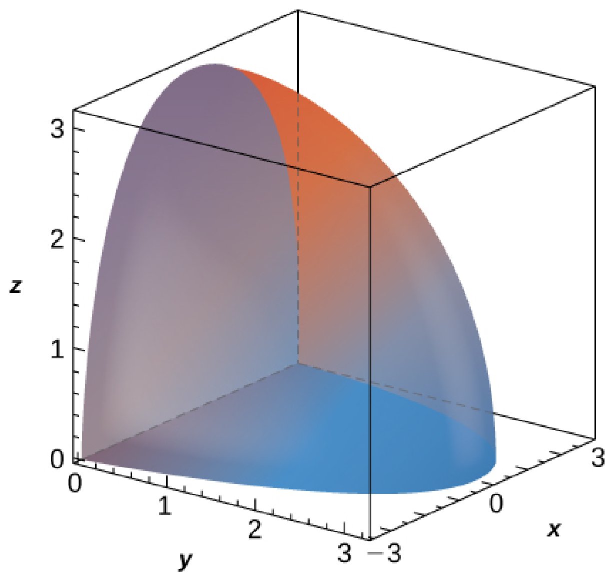


Solution:

$180\pi\sqrt{10}$

Exercise:

Problem: $f(x,y,z)=1-\sqrt{x^2+y^2+z^2}$, $B=\left\{(x,y,z) \middle| x^2+y^2+z^2 \le 9, y \ge 0, z \ge 0\right\}$



Exercise:

Problem:

$f(x, y, z) = \sqrt{x^2 + y^2}$, B is bounded above by the half-sphere $x^2 + y^2 + z^2 = 9$ with $z \geq 0$ and below by the cone $2z^2 = x^2 + y^2$.

Solution:

$\frac{81\pi(\pi-2)}{16}$

Exercise:

Problem:

$f(x, y, z) = z$, B is bounded above by the half-sphere $x^2 + y^2 + z^2 = 16$ with $z \geq 0$ and below by the cone $2z^2 = x^2 + y^2$.

Exercise:

Problem:

Show that if $F(\rho, \theta, \varphi) = f(\rho)g(\theta)h(\varphi)$ is a continuous function on the spherical box $B = \{(\rho, \theta, \varphi) | a \leq \rho \leq b, \alpha \leq \theta \leq \beta, \gamma \leq \varphi \leq \psi\}$, then

Equation:

$$\iiint_B F\, dV = \left(\int_a^b \rho^2 f(\rho) dr\right) \left(\int_\alpha^\beta g(\theta) d\theta\right) \left(\int_\gamma^\psi h(\varphi) \sin\varphi\, d\varphi\right).$$

Exercise:

Problem:

a. A function F is said to have spherical symmetry if it depends on the distance to the origin only, that is, it can be expressed in spherical coordinates as $F(x, y, z) = f(\rho)$, where

$\rho = \sqrt{x^2 + y^2 + z^2}$. Show that

Equation:

$$\iiint_B F(x, y, z) dV = 2\pi \int_a^b \rho^2 f(\rho) d\rho,$$

where B is the region between the upper concentric hemispheres of radii a and b centered at the origin, with $0 < a < b$ and F a spherical function defined on B.

b. Use the previous result to show that $\iiint_B \left(x^2 + y^2 + z^2\right)\sqrt{x^2 + y^2 + z^2}\, dV = 21\pi$, where

Equation:

$$B = \left\{(x, y, z)\middle| 1 \le x^2 + y^2 + z^2 \le 2, z \ge 0\right\}.$$

Exercise:

Problem:

a. Let B be the region between the upper concentric hemispheres of radii a and b centered at the origin and situated in the first octant, where $0 < a < b$. Consider F a function defined on B whose form in spherical coordinates (ρ, θ, φ) is $F\,(x, y, z) = f(\rho)\cos\varphi$. Show that if

$g(a) = g(b) = 0$ and $\int_a^b h(\rho) d\rho = 0$, then

Equation:

$$\iiint_B F(x, y, z) dV = \frac{\pi^2}{4}[ah(a) - bh(b)],$$

where g is an antiderivative of f and h is an antiderivative of g.

b. Use the previous result to show that $\iiint_B \frac{z \cos\sqrt{x^2 + y^2 + z^2}}{\sqrt{x^2 + y^2 + z^2}} dV = \frac{3\pi^2}{2}$, where B is the region between the upper concentric hemispheres of radii π and 2π centered at the origin and situated in the first octant.

In the following exercises, the function f and region E are given.

a. Express the region E and function f in cylindrical coordinates.
b. Convert the integral $\iiint_B f\,(x, y, z) dV$ into cylindrical coordinates and evaluate it.

Exercise:

Problem: $f(x, y, z) = z$; $E = \left\{(x, y, z) \middle| 0 \le x^2 + y^2 + z^2 \le 1, z \ge 0\right\}$

Exercise:

Problem: $f(x, y, z) = x + y$; $E = \left\{(x, y, z) \middle| 1 \le x^2 + y^2 + z^2 \le 2, z \ge 0, y \ge 0\right\}$

Solution:

a. $f(\rho, \theta, \varphi) = \rho \sin \varphi (\cos \theta + \sin \theta)$, $E = \left\{(\rho, \theta, \varphi) \middle| 1 \le \rho \le 2, 0 \le \theta \le \pi, 0 \le \varphi \le \frac{\pi}{2}\right\}$; b. $\int_0^{\pi} \int_0^{\pi/2} \int_1^2 \rho^3 \cos \varphi \sin \varphi \, d\rho \, d\varphi \, d\theta = \frac{15\pi}{8}$

Exercise:

Problem: $f(x, y, z) = 2xy$; $E = \left\{(x, y, z) \middle| \sqrt{x^2 + y^2} \le z \le \sqrt{1 - x^2 - y^2}, x \ge 0, y \ge 0\right\}$

Exercise:

Problem: $f(x, y, z) = z$; $E = \left\{(x, y, z) \middle| x^2 + y^2 + z^2 - 2z \le 0, \sqrt{x^2 + y^2} \le z\right\}$

Solution:

a. $f(\rho, \theta, \varphi) = \rho \cos \varphi$; $E = \left\{(\rho, \theta, \varphi) \middle| 0 \le \rho \le 2 \cos \varphi, 0 \le \theta \le \frac{\pi}{2}, 0 \le \varphi \le \frac{\pi}{4}\right\}$; b. $\int_0^{\pi/2} \int_0^{\pi/4} \int_0^{2 \cos \varphi} \rho^3 \sin \varphi \cos \varphi \, d\rho \, d\varphi \, d\theta = \frac{7\pi}{24}$

In the following exercises, find the volume of the solid E whose boundaries are given in rectangular coordinates.

Exercise:

Problem: $E = \left\{(x, y, z) \middle| \sqrt{x^2 + y^2} \le z \le \sqrt{16 - x^2 - y^2}, x \ge 0, y \ge 0\right\}$

Exercise:

Problem: $E = \left\{(x, y, z) \middle| x^2 + y^2 + z^2 - 2z \le 0, \sqrt{x^2 + y^2} \le z\right\}$

Solution:

$\frac{\pi}{4}$

Exercise:

Problem:

Use spherical coordinates to find the volume of the solid situated outside the sphere $\rho = 1$ and inside the sphere $\rho = \cos \varphi$, with $\varphi \in \left[0, \frac{\pi}{2}\right]$.

Exercise:

Problem:

Use spherical coordinates to find the volume of the ball $\rho \le 3$ that is situated between the cones $\varphi = \frac{\pi}{4}$ and $\varphi = \frac{\pi}{3}$.

Solution:

$$9\pi\left(\sqrt{2} - 1\right)$$

Exercise:

Problem:

Convert the integral $\int_{-4}^{4} \int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} \int_{-\sqrt{16-x^2-y^2}}^{\sqrt{16-x^2-y^2}} \left(x^2 + y^2 + z^2\right) dz\, dx\, dy$ into an integral in spherical coordinates.

Exercise:

Problem:

Convert the integral $\int_{0}^{4} \int_{0}^{\sqrt{16-x^2}} \int_{-\sqrt{16-x^2-y^2}}^{\sqrt{16-x^2-y^2}} \left(x^2 + y^2 + z^2\right)^2 dz\, dy\, dx$ into an integral in spherical coordinates.

Solution:

$$\int_{0}^{\pi/2} \int_{0}^{\pi/2} \int_{0}^{4} \rho^6 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

Exercise:

Problem:

Convert the integral $\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{16-x^2-y^2}} dz\, dy\, dx$ into an integral in spherical coordinates and evaluate it.

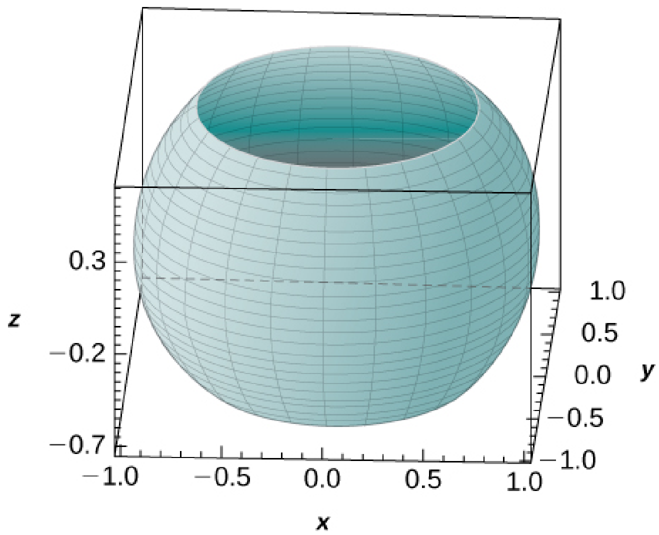
Exercise:

Problem:

[T] Use a CAS to graph the solid whose volume is given by the iterated integral in spherical coordinates $\int_{\pi/2}^{\pi} \int_{5\pi/6}^{\pi/6} \int_{0}^{2} \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$. Find the volume V of the solid. Round your answer to three decimal places.

Solution:

$V = \frac{4\pi\sqrt{3}}{3} \approx 7.255$



Exercise:

Problem:

[T] Use a CAS to graph the solid whose volume is given by the iterated integral in spherical coordinates as $\int_0^{2\pi} \int_{3\pi/4}^{\pi/4} \int_0^1 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$. Find the volume V of the solid. Round your answer to three decimal places.

Exercise:

Problem:

[T] Use a CAS to evaluate the integral $\iiint_E (x^2 + y^2) dV$ where E lies above the paraboloid $z = x^2 + y^2$ and below the plane $z = 3y$.

Solution:

$\frac{343\pi}{32}$

Exercise:

Problem: [T]

a. Evaluate the integral $\iiint_E e^{\sqrt{x^2+y^2+z^2}} dV$, where E is bounded by the spheres $4x^2 + 4y^2 + 4z^2 = 1$ and $x^2 + y^2 + z^2 = 1$.

b. Use a CAS to find an approximation of the previous integral. Round your answer to two decimal places.

Exercise:

Problem:

Express the volume of the solid inside the sphere $x^2 + y^2 + z^2 = 16$ and outside the cylinder $x^2 + y^2 = 4$ as triple integrals in cylindrical coordinates and spherical coordinates, respectively.

Solution:

$$\int_0^{2\pi} \int_2^4 \int_{-\sqrt{16-r^2}}^{\sqrt{16-r^2}} r\, dz\, dr\, d\theta; \int_{\pi/6}^{5\pi/6} \int_0^{2\pi} \int_{2\csc\varphi}^{4} \rho^2 \sin\rho\, d\rho\, d\theta\, d\varphi$$

Exercise:

Problem:

Express the volume of the solid inside the sphere $x^2 + y^2 + z^2 = 16$ and outside the cylinder $x^2 + y^2 = 4$ that is located in the first octant as triple integrals in cylindrical coordinates and spherical coordinates, respectively.

Exercise:

Problem:

The power emitted by an antenna has a power density per unit volume given in spherical coordinates by

$$p(\rho, \theta, \varphi) = \frac{P_0}{\rho^2} \cos^2\theta \sin^4\varphi,$$

where P_0 is a constant with units in watts. The total power within a sphere B of radius r meters is defined as $P = \iiint_B p(\rho, \theta, \varphi) dV$. Find the total power P.

Solution:

$P = \frac{32 P_0 \pi}{3}$ watts

Exercise:

Problem:

Use the preceding exercise to find the total power within a sphere B of radius 5 meters when the power density per unit volume is given by $p(\rho, \theta, \varphi) = \frac{30}{\rho^2} \cos^2\theta \sin^4\varphi$.

Exercise:

Problem:

A charge cloud contained in a sphere B of radius r centimeters centered at the origin has its charge density given by $q\left(x,y,z\right)=k\sqrt{x^2+y^2+z^2}\,\frac{\mu\,\mathrm{C}}{\mathrm{cm}^3}$, where $k>0$. The total charge contained in B is given by $Q=\iiint\limits_B q\left(x,y,z\right)dV$. Find the total charge Q.

Solution:

$Q=kr^4\pi\mu C$

Exercise:

Problem:

Use the preceding exercise to find the total charge cloud contained in the unit sphere if the charge density is $q\left(x,y,z\right)=20\sqrt{x^2+y^2+z^2}\,\frac{\mu\,\mathrm{C}}{\mathrm{cm}^3}$.

Glossary

triple integral in cylindrical coordinates

the limit of a triple Riemann sum, provided the following limit exists:

Equation:

$$\lim_{l,m,n\to\infty}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n}f(r_{ijk}^{*},\theta_{ijk}^{*},z_{ijk}^{*})r_{ijk}^{*}\Delta r\Delta\theta\Delta z$$

triple integral in spherical coordinates

the limit of a triple Riemann sum, provided the following limit exists:

Equation:

$$\lim_{l,m,n\to\infty}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n}f(\rho_{ijk}^{*},\theta_{ijk}^{*},\varphi_{ijk}^{*})(\rho_{ijk}^{*})^2\sin\varphi\Delta\rho\Delta\theta\Delta\varphi$$

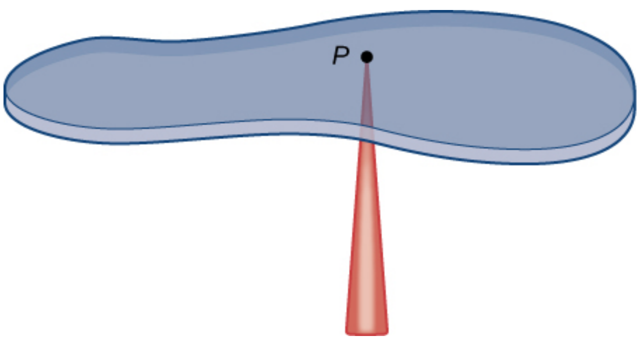
Calculating Centers of Mass and Moments of Inertia

- Use double integrals to locate the center of mass of a two-dimensional object.
- Use double integrals to find the moment of inertia of a two-dimensional object.
- Use triple integrals to locate the center of mass of a three-dimensional object.

We have already discussed a few applications of multiple integrals, such as finding areas, volumes, and the average value of a function over a bounded region. In this section we develop computational techniques for finding the center of mass and moments of inertia of several types of physical objects, using double integrals for a lamina (flat plate) and triple integrals for a three-dimensional object with variable density. The density is usually considered to be a constant number when the lamina or the object is homogeneous; that is, the object has uniform density.

Center of Mass in Two Dimensions

The center of mass is also known as the center of gravity if the object is in a uniform gravitational field. If the object has uniform density, the center of mass is the geometric center of the object, which is called the centroid. [link] shows a point P as the center of mass of a lamina. The lamina is perfectly balanced about its center of mass.



A lamina is perfectly balanced on a spindle if the lamina's center of mass sits on the spindle.

To find the coordinates of the center of mass $P(\bar{x}, \bar{y})$ of a lamina, we need to find the moment M_x of the lamina about the x-axis and the moment M_y about the y-axis. We also need to find the mass m of the lamina. Then

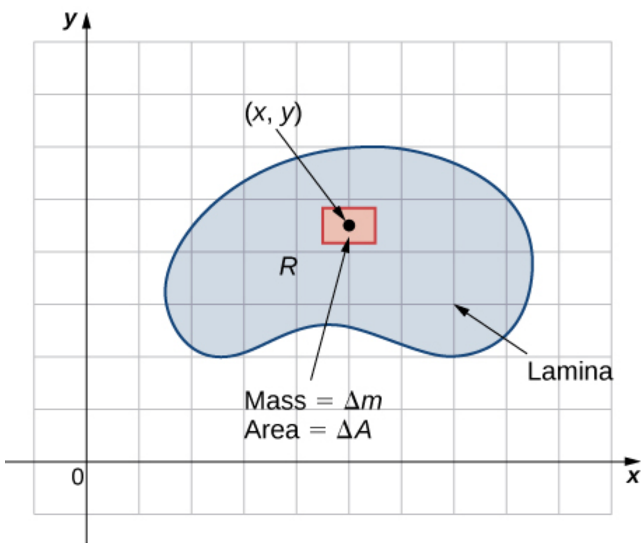
Equation:

$$\bar{x} = \frac{M_y}{m} \text{ and } \bar{y} = \frac{M_x}{m}.$$

Refer to Moments and Centers of Mass for the definitions and the methods of single integration to find the center of mass of a one-dimensional object (for example, a thin rod). We are going to use a similar idea here except that the object is a two-dimensional lamina and we use a double integral.

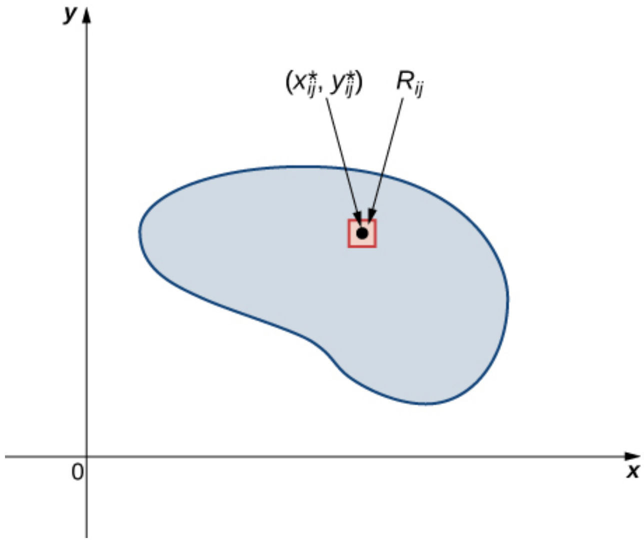
If we allow a constant density function, then $\bar{x} = \frac{M_y}{m}$ and $\bar{y} = \frac{M_x}{m}$ give the *centroid* of the lamina.

Suppose that the lamina occupies a region R in the xy-plane, and let $\rho(x, y)$ be its density (in units of mass per unit area) at any point (x, y). Hence, $\rho(x, y) = \lim_{\Delta A \to 0} \frac{\Delta m}{\Delta A}$, where Δm and ΔA are the mass and area of a small rectangle containing the point (x, y) and the limit is taken as the dimensions of the rectangle go to 0 (see the following figure).



The density of a lamina at a point is the limit of its mass per area in a small rectangle about the point as the area goes to zero.

Just as before, we divide the region R into tiny rectangles R_{ij} with area ΔA and choose $\left(x_{ij}^*, y_{ij}^*\right)$ as sample points. Then the mass m_{ij} of each R_{ij} is equal to $\rho\left(x_{ij}^*, y_{ij}^*\right)\Delta A$ ([link]). Let k and l be the number of subintervals in x and y, respectively. Also, note that the shape might not always be rectangular but the limit works anyway, as seen in previous sections.



Subdividing the lamina into tiny rectangles R_{ij}, each containing a sample point (x_{ij}^*, y_{ij}^*).

Hence, the mass of the lamina is

Equation:

$$m = \lim_{k,l \to \infty} \sum_{i=1}^{k} \sum_{j=1}^{l} m_{ij} = \lim_{k,l \to \infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*) \Delta A = \iint_R \rho(x, y) dA.$$

Let's see an example now of finding the total mass of a triangular lamina.

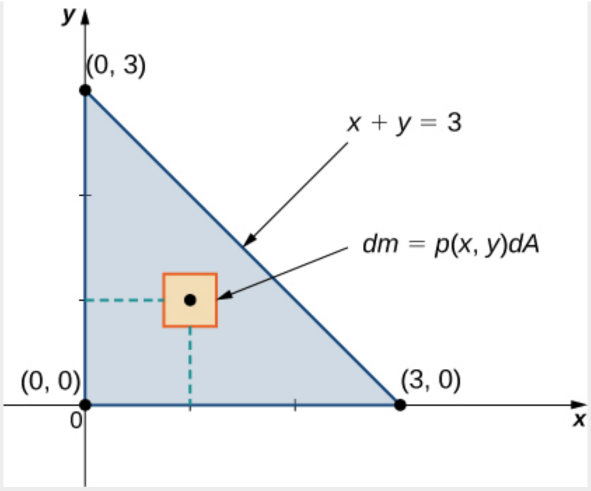
Example:
Exercise:

Problem:
Finding the Total Mass of a Lamina

Consider a triangular lamina R with vertices $(0, 0)$, $(0, 3)$, $(3, 0)$ and with density $\rho(x, y) = xy\,\text{kg/m}^2$. Find the total mass.

Solution:

A sketch of the region R is always helpful, as shown in the following figure.



A lamina in the xy-plane with density $\rho(x, y) = xy$.

Using the expression developed for mass, we see that

Equation:

$$\begin{aligned} m &= \iint_R dm = \iint_R \rho(x,y)dA = \int_{x=0}^{x=3} \int_{y=0}^{y=3-x} xy\,dy\,dx = \int_{x=0}^{x=3} \left[x \frac{y^2}{2} \bigg|_{y=0}^{y=3-x} \right] dx \\ &= \int_{x=0}^{x=3} \frac{1}{2} x(3-x)^2 dx = \left[\frac{9x^2}{4} - x^3 + \frac{x^4}{8} \right] \bigg|_{x=0}^{x=3} \\ &= \frac{27}{8}. \end{aligned}$$

The computation is straightforward, giving the answer $m = \frac{27}{8}$ kg.

Note:
Exercise:

Problem:

Consider the same region R as in the previous example, and use the density function $\rho(x, y) = \sqrt{xy}$. Find the total mass.

Solution:

$\frac{9\pi}{8}$ kg

Now that we have established the expression for mass, we have the tools we need for calculating moments and centers of mass. The moment M_x about the x-axis for R is the limit of the sums of moments of the regions R_{ij} about the x-axis. Hence

Equation:

$$M_x = \lim_{k,l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \left(y_{ij}^{*}\right) m_{ij} = \lim_{k,l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \left(y_{ij}^{*}\right) \rho\left(x_{ij}^{*}, y_{ij}^{*}\right) \Delta A = \iint_R y\rho\left(x, y\right) dA.$$

Similarly, the moment M_y about the y-axis for R is the limit of the sums of moments of the regions R_{ij} about the y-axis. Hence

Equation:

$$M_y = \lim_{k,l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \left(x_{ij}^{*}\right) m_{ij} = \lim_{k,l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \left(y_{ij}^{*}\right) \rho\left(x_{ij}^{*}, y_{ij}^{*}\right) \Delta A = \iint_R x\rho\left(x, y\right) dA.$$

Example:

Exercise:

Problem:

Finding Moments

Consider the same triangular lamina R with vertices $(0, 0), (0, 3),\ (3, 0)$ and with density $\rho\left(x, y\right) = xy$. Find the moments M_x and M_y.

Solution:

Use double integrals for each moment and compute their values:

Equation:

$$M_x = \iint_R y\rho\left(x, y\right) dA = \int_{x=0}^{x=3} \int_{y=0}^{y=3-x} xy^2 \, dy \, dx = \frac{81}{20},$$

Equation:

$$M_y = \iint_R x\rho\left(x, y\right) dA = \int_{x=0}^{x=3} \int_{y=0}^{y=3-x} x^2 \, yd\, y \, dx = \frac{81}{20}.$$

The computation is quite straightforward.

Note:

Exercise:

Problem:

Consider the same lamina R as above, and use the density function $\rho(x, y) = \sqrt{xy}$. Find the moments M_x and M_y.

Solution:

$M_x = \frac{81\pi}{64}$ and $M_y = \frac{81\pi}{64}$

Finally we are ready to restate the expressions for the center of mass in terms of integrals. We denote the *x*-coordinate of the center of mass by $\bar{x}$ and the *y*-coordinate by $\bar{y}$. Specifically,
Equation:

$$\bar{x} = \frac{M_y}{m} = \frac{\iint\limits_R x\rho(x, y)dA}{\iint\limits_R \rho(x, y)dA} \text{ and } \bar{y} = \frac{M_x}{m} \frac{\iint\limits_R y\rho(x, y)dA}{\iint\limits_R \rho(x, y)dA}.$$

Example:
Exercise:

Problem:
Finding the Center of Mass

Again consider the same triangular region R with vertices $(0, 0)$, $(0, 3)$, $(3, 0)$ and with density function $\rho(x, y) = xy$. Find the center of mass.

Solution:

Using the formulas we developed, we have
Equation:

$$\bar{x} = \frac{M_y}{m} = \frac{\iint\limits_R x\rho(x, y)dA}{\iint\limits_R \rho(x, y)dA} = \frac{81/20}{27/8} = \frac{6}{5},$$

Equation:

$$\bar{y} = \frac{M_x}{m} = \frac{\iint\limits_R y\rho(x, y)dA}{\iint\limits_R \rho(x, y)dA} = \frac{81/20}{27/8} = \frac{6}{5}.$$

Therefore, the center of mass is the point $\left(\frac{6}{5}, \frac{6}{5}\right)$.

Analysis

If we choose the density $\rho\left(x, y\right)$ instead to be uniform throughout the region (i.e., constant), such as the value 1 (any constant will do), then we can compute the centroid,

Equation:

$$x_c = \frac{M_y}{m} = \frac{\iint\limits_R x\,dA}{\iint\limits_R dA} = \frac{9/2}{9/2} = 1,$$

$$y_c = \frac{M_x}{m} \frac{\iint\limits_R y\,dA}{\iint\limits_R dA} = \frac{9/2}{9/2} = 1.$$

Notice that the center of mass $\left(\frac{6}{5}, \frac{6}{5}\right)$ is not exactly the same as the centroid $\left(1, 1\right)$ of the triangular region. This is due to the variable density of R. If the density is constant, then we just use $\rho\left(x, y\right) = c$ (constant). This value cancels out from the formulas, so for a constant density, the center of mass coincides with the centroid of the lamina.

Note:

Exercise:

Problem:

Again use the same region R as above and the density function $\rho\left(x, y\right) = \sqrt{xy}$. Find the center of mass.

Solution:

$\overline{x} = \frac{M_y}{m} = \frac{81\pi/64}{9\pi/8} = \frac{9}{8}$ and $\overline{y} = \frac{M_x}{m} = \frac{81\pi/64}{9\pi/8} = \frac{9}{8}$.

Once again, based on the comments at the end of [link], we have expressions for the centroid of a region on the plane:

Equation:

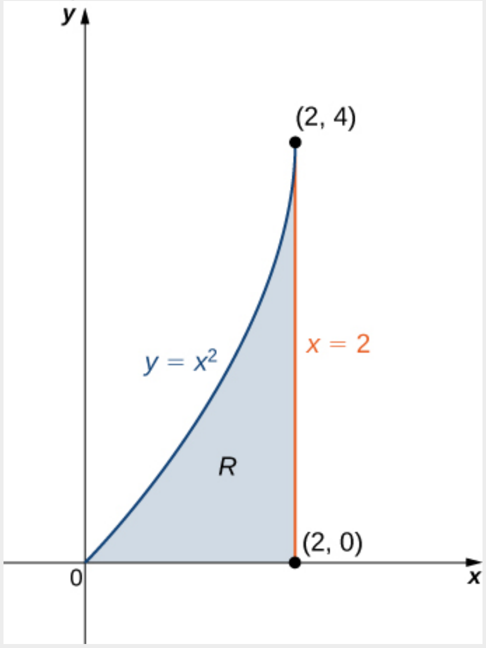
$$x_c = \frac{M_y}{m} = \frac{\iint\limits_R x\,dA}{\iint\limits_R dA} \text{ and } y_c = \frac{M_x}{m} \frac{\iint\limits_R y\,dA}{\iint\limits_R dA}.$$

We should use these formulas and verify the centroid of the triangular region R referred to in the last three examples.

Example:
Exercise:

Problem:
Finding Mass, Moments, and Center of Mass

Find the mass, moments, and the center of mass of the lamina of density $\rho(x, y) = x + y$ occupying the region R under the curve $y = x^2$ in the interval $0 \le x \le 2$ (see the following figure).



Locating the center of mass of a lamina R with density $\rho(x, y) = x + y$.

Solution:

First we compute the mass m. We need to describe the region between the graph of $y = x^2$ and the vertical lines $x = 0$ and $x = 2$:

Equation:

$$\begin{aligned} m &= \iint_R dm = \iint_R \rho(x, y) dA = \int_{x=0}^{x=2} \int_{y=0}^{y=x^2} (x + y) dy\, dx = \int_{x=0}^{x=2} \left[xy + \frac{y^2}{2} \Bigg|_{y=0}^{y=x^2} \right] dx \\ &= \int_{x=0}^{x=2} \left[x^3 + \frac{x^4}{2} \right] dx = \left[\tfrac{x^4}{4} + \tfrac{x^5}{10} \right] \Bigg|_{x=0}^{x=2} = \tfrac{36}{5}. \end{aligned}$$

Now compute the moments M_x and M_y:

Equation:

$$M_x = \iint_R y\rho(x,y)dA = \int_{x=0}^{x=2} \int_{y=0}^{y=x^2} y(x+y)dy\,dx = \frac{80}{7},$$

Equation:

$$M_y = \iint_R x\rho(x,y)dA = \int_{x=0}^{x=2} \int_{y=0}^{y=x^2} x(x+y)dy\,dx = \frac{176}{15}.$$

Finally, evaluate the center of mass,

Equation:

$$\bar{x} = \frac{M_y}{m} = \frac{\iint_R x\rho(x,y)dA}{\iint_R \rho(x,y)dA} = \frac{176/15}{36/5} = \frac{44}{27},$$

$$\bar{y} = \frac{M_x}{m} = \frac{\iint_R y\rho(x,y)dA}{\iint_R \rho(x,y)dA} = \frac{80/7}{36/5} = \frac{100}{63}.$$

Hence the center of mass is $(\bar{x}, \bar{y}) = \left(\frac{44}{27}, \frac{100}{63}\right)$.

Note:
Exercise:

Problem:

Calculate the mass, moments, and the center of mass of the region between the curves $y = x$ and $y = x^2$ with the density function $\rho(x,y) = x$ in the interval $0 \le x \le 1$.

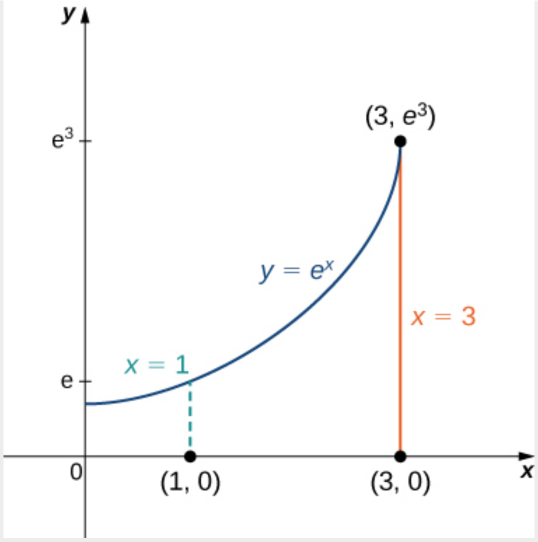
Solution:

$\bar{x} = \frac{M_y}{m} = \frac{1/20}{1/12} = \frac{3}{5}$ and $\bar{y} = \frac{M_x}{m} = \frac{1/24}{1/12} = \frac{1}{2}$

Example:
Exercise:

Problem:
Finding a Centroid

Find the centroid of the region under the curve $y = e^x$ over the interval $1 \le x \le 3$ (see the following figure).



Finding a centroid of a region below the curve $y = e^x$.

Solution:

To compute the centroid, we assume that the density function is constant and hence it cancels out:
Equation:

$$x_c = \frac{M_y}{m} = \frac{\iint\limits_R x\,dA}{\iint\limits_R dA} \text{ and } y_c = \frac{M_x}{m} = \frac{\iint\limits_R y\,dA}{\iint\limits_R dA},$$

$$x_c = \frac{M_y}{m} = \frac{\iint\limits_R x\,dA}{\iint\limits_R dA} = \frac{\int_{x=1}^{x=3}\int_{y=0}^{y=e^x} x\,dy\,dx}{\int_{x=1}^{x=3}\int_{y=0}^{y=e^x} dy\,dx} = \frac{\int_{x=1}^{x=3} xe^x\,dx}{\int_{x=1}^{x=3} e^x\,dx} = \frac{2e^3}{e^3-e} = \frac{2e^2}{e^2-1},$$

$$y_c = \frac{M_x}{m} = \frac{\iint\limits_R y\,dA}{\iint\limits_R dA} = \frac{\int_{x=1}^{x=3}\int_{y=0}^{y=e^x} y\,dy\,dx}{\int_{x=1}^{x=3}\int_{y=0}^{y=e^x} dy\,dx} = \frac{\int_{x=1}^{x=3} \frac{e^{2x}}{2}\,dx}{\int_{x=1}^{x=3} e^x\,dx} = \frac{\frac{1}{4}e^2(e^4-1)}{e(e^2-1)} = \frac{1}{4}e\left(e^2+1\right).$$

Thus the centroid of the region is
Equation:

$$(x_c, y_c) = \left(\frac{2e^2}{e^2-1}, \frac{1}{4}e\left(e^2+1\right)\right).$$

Note:
Exercise:

Problem:

Calculate the centroid of the region between the curves $y = x$ and $y = \sqrt{x}$ with uniform density in the interval $0 \le x \le 1$.

Solution:

$x_c = \frac{M_y}{m} = \frac{1/15}{1/6} = \frac{2}{5}$ and $y_c = \frac{M_x}{m} = \frac{1/12}{1/6} = \frac{1}{2}$

Moments of Inertia

For a clear understanding of how to calculate moments of inertia using double integrals, we need to go back to the general definition in Section 6.6. The moment of inertia of a particle of mass m about an axis is mr^2, where r is the distance of the particle from the axis. We can see from [link] that the moment of inertia of the subrectangle R_{ij} about the x-axis is $(y_{ij}^*)^2\rho(x_{ij}^*, y_{ij}^*)\Delta A$. Similarly, the moment of inertia of the subrectangle R_{ij} about the y-axis is $(x_{ij}^*)^2\rho(x_{ij}^*, y_{ij}^*)\Delta A$. The moment of inertia is related to the rotation of the mass; specifically, it measures the tendency of the mass to resist a change in rotational motion about an axis.

The moment of inertia I_x about the x-axis for the region R is the limit of the sum of moments of inertia of the regions R_{ij} about the x-axis. Hence
Equation:

$$I_x = \lim_{k,l\to\infty}\sum_{i=1}^{k}\sum_{j=1}^{l}\left(y_{ij}^*\right)^2 m_{ij} = \lim_{k,l\to\infty}\sum_{i=1}^{k}\sum_{j=1}^{l}\left(y_{ij}^*\right)^2 \rho\left(x_{ij}^*, y_{ij}^*\right)\Delta A = \iint\limits_R y^2\rho(x,y)dA.$$

Similarly, the moment of inertia I_y about the y-axis for R is the limit of the sum of moments of inertia of the regions R_{ij} about the y-axis. Hence
Equation:

$$I_y = \lim_{k,l\to\infty}\sum_{i=1}^{k}\sum_{j=1}^{l}\left(x_{ij}^*\right)^2 m_{ij} = \lim_{k,l\to\infty}\sum_{i=1}^{k}\sum_{j=1}^{l}\left(x_{ij}^*\right)^2 \rho\left(x_{ij}^*, y_{ij}^*\right)\Delta A = \iint\limits_R x^2\rho(x,y)dA.$$

Sometimes, we need to find the moment of inertia of an object about the origin, which is known as the polar moment of inertia. We denote this by I_0 and obtain it by adding the moments of inertia I_x and I_y. Hence
Equation:

$$I_0 = I_x + I_y = \iint\limits_R (x^2 + y^2)\rho(x,y)dA.$$

All these expressions can be written in polar coordinates by substituting $x = r\cos\theta$, $y = r\sin\theta$, and $dA = r\,dr\,d\theta$. For example, $I_0 = \iint\limits_R r^2 \rho\,(r\cos\theta, r\sin\theta) dA$.

Example:
Exercise:

Problem:
Finding Moments of Inertia for a Triangular Lamina

Use the triangular region R with vertices $(0,0)$, $(2,2)$, and $(2,0)$ and with density $\rho\,(x,y) = xy$ as in previous examples. Find the moments of inertia.

Solution:

Using the expressions established above for the moments of inertia, we have
Equation:

$$\begin{aligned} I_x &= \iint\limits_R y^2 \rho\,(x,y) dA = \int\limits_{x=0}^{x=2} \int\limits_{y=0}^{y=x} xy^3 dy\,dx = \tfrac{8}{3}, \\ I_y &= \iint\limits_R x^2 \rho\,(x,y) dA = \int\limits_{x=0}^{x=2} \int\limits_{y=0}^{y=x} x^3 y\,dy\,dx = \tfrac{16}{3}, \\ I_0 &= \iint\limits_R \left(x^2 + y^2\right) \rho(x,y) dA = \int\limits_0^2 \int\limits_0^x \left(x^2 + y^2\right) xy\,dy\,dx \\ &= I_x + I_y = 8. \end{aligned}$$

Note:
Exercise:

Problem:

Again use the same region R as above and the density function $\rho\,(x,y) = \sqrt{xy}$. Find the moments of inertia.

Solution:

$I_x = \int\limits_{x=0}^{x=2} \int\limits_{y=0}^{y=x} y^2 \sqrt{xy}\,dy\,dx = \frac{64}{35}$ and $I_y = \int\limits_{x=0}^{x=2} \int\limits_{y=0}^{y=x} x^2 \sqrt{xy}\,dy\,dx = \frac{64}{35}$. Also, $I_0 = \int\limits_{x=0}^{x=2} \int\limits_{y=0}^{y=x} \left(x^2 + y^2\right) \sqrt{xy}\,dy\,dx = \frac{128}{21}$.

As mentioned earlier, the moment of inertia of a particle of mass m about an axis is mr^2 where r is the distance of the particle from the axis, also known as the **radius of gyration**.

Hence the radii of gyration with respect to the x-axis, the y-axis, and the origin are

Equation:

$$R_x = \sqrt{\frac{I_x}{m}}, R_y = \sqrt{\frac{I_y}{m}}, \text{and } R_0 = \sqrt{\frac{I_0}{m}},$$

respectively. In each case, the radius of gyration tells us how far (perpendicular distance) from the axis of rotation the entire mass of an object might be concentrated. The moments of an object are useful for finding information on the balance and torque of the object about an axis, but radii of gyration are used to describe the distribution of mass around its centroidal axis. There are many applications in engineering and physics. Sometimes it is necessary to find the radius of gyration, as in the next example.

Example:

Exercise:

Problem:

Finding the Radius of Gyration for a Triangular Lamina

Consider the same triangular lamina R with vertices $(0, 0)$, $(2, 2)$, and $(2, 0)$ and with density $\rho(x, y) = xy$ as in previous examples. Find the radii of gyration with respect to the x-axis, the y-axis, and the origin.

Solution:

If we compute the mass of this region we find that $m = 2$. We found the moments of inertia of this lamina in [link]. From these data, the radii of gyration with respect to the x-axis, y-axis, and the origin are, respectively,

Equation:

$$\begin{aligned} R_x &= \sqrt{\frac{I_x}{m}} = \sqrt{\frac{8/3}{2}} = \sqrt{\frac{8}{6}} = \frac{2\sqrt{3}}{3}, \\ R_y &= \sqrt{\frac{I_y}{m}} = \sqrt{\frac{16/3}{2}} = \sqrt{\frac{8}{3}} = \frac{2\sqrt{6}}{3}, \\ R_0 &= \sqrt{\frac{I_0}{m}} = \sqrt{\frac{8}{2}} = \sqrt{4} = 2. \end{aligned}$$

Note:

Exercise:

Problem:

Use the same region R from [link] and the density function $\rho(x, y) = \sqrt{xy}$. Find the radii of gyration with respect to the x-axis, the y-axis, and the origin.

Solution:

$R_x = \frac{6\sqrt{35}}{35}$, $R_y = \frac{6\sqrt{15}}{15}$, and $R_0 = \frac{4\sqrt{42}}{7}$.

Hint

Follow the steps shown in the previous example.

Center of Mass and Moments of Inertia in Three Dimensions

All the expressions of double integrals discussed so far can be modified to become triple integrals.

Note:
Definition
If we have a solid object Q with a density function $\rho(x, y, z)$ at any point (x, y, z) in space, then its mass is
Equation:

$$m = \iiint\limits_Q \rho(x, y, z)dV.$$

Its moments about the xy-plane, the xz-plane, and the yz-plane are
Equation:

$$M_{xy} = \iiint\limits_Q z\rho(x, y, z)dV,\ M_{xz} = \iiint\limits_Q y\rho(x, y, z)dV,$$
$$M_{yz} = \iiint\limits_Q x\rho(x, y, z)dV.$$

If the center of mass of the object is the point $\left(\bar{x}, \bar{y}, \bar{z}\right)$, then
Equation:

$$\bar{x} = \frac{M_{yz}}{m},\ \bar{y} = \frac{M_{xz}}{m}, \bar{z} = \frac{M_{xy}}{m}.$$

Also, if the solid object is homogeneous (with constant density), then the center of mass becomes the centroid of the solid. Finally, the moments of inertia about the yz-plane, the xz-plane, and the xy-plane are
Equation:

$$I_x = \iiint\limits_Q \left(y^2 + z^2\right)\rho(x, y, z)dV,$$
$$I_y = \iiint\limits_Q \left(x^2 + z^2\right)\rho(x, y, z)dV,$$
$$I_z = \iiint\limits_Q \left(x^2 + y^2\right)\rho(x, y, z)dV.$$

Example:
Exercise:

Problem:
Finding the Mass of a Solid

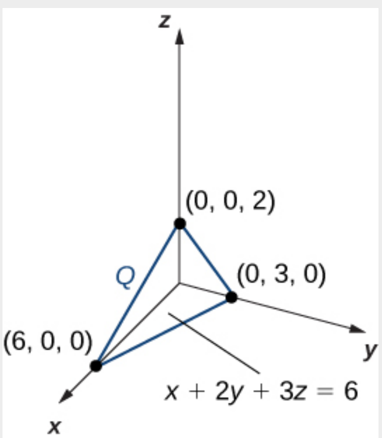
Suppose that Q is a solid region bounded by $x+2y+3z=6$ and the coordinate planes and has density $\rho(x,y,z)=x^2yz$. Find the total mass.

Solution:

The region Q is a tetrahedron ([link]) meeting the axes at the points $(6,0,0)$, $(0,3,0)$, and $(0,0,2)$. To find the limits of integration, let $z=0$ in the slanted plane $z=\frac{1}{3}(6-x-2y)$. Then for x and y find the projection of Q onto the xy-plane, which is bounded by the axes and the line $x+2y=6$. Hence the mass is

Equation:

$$m=\iiint\limits_{Q}\rho(x,y,z)dV=\int_{x=0}^{x=6}\int_{y=0}^{y=1/2(6-x)}\int_{z=0}^{z=1/3(6-x-2y)}x^2yz\,dz\,dy\,dx=\frac{108}{35}\approx 3.086.$$



Finding the mass of a three-dimensional solid Q.

Note:
Exercise:

Problem:

Consider the same region Q ([link]), and use the density function $\rho(x,y,z)=xy^2z$. Find the mass.

Solution:

$\frac{54}{35}=1.543$

Hint

Follow the steps in the previous example.

Example:
Exercise:

Problem:
Finding the Center of Mass of a Solid

Suppose Q is a solid region bounded by the plane $x+2y+3z=6$ and the coordinate planes with density $\rho\left(x,y,z\right)=x^2yz$ (see [link]). Find the center of mass using decimal approximation.

Solution:

We have used this tetrahedron before and know the limits of integration, so we can proceed to the computations right away. First, we need to find the moments about the xy-plane, the xz-plane, and the yz-plane:
Equation:

$$M_{xy}=\iiint_Q z\rho\left(x,y,z\right)dV=\int_{x=0}^{x=6}\int_{y=0}^{y=1/2(6-x)}\int_{z=0}^{z=1/3(6-x-2y)} x^2yz^2dz\,dy\,dx=\frac{54}{35}\approx 1.543,$$
$$M_{xz}=\iiint_Q y\rho\left(x,y,z\right)dV=\int_{x=0}^{x=6}\int_{y=0}^{y=1/2(6-x)}\int_{z=0}^{z=1/3(6-x-2y)} x^2y^2z\,dz\,dy\,dx=\frac{81}{35}\approx 2.314,$$
$$M_{yz}=\iiint_Q x\rho\left(x,y,z\right)dV=\int_{x=0}^{x=6}\int_{y=0}^{y=1/2(6-x)}\int_{z=0}^{z=1/3(6-x-2y)} x^3yz\,dz\,dy\,dx=\frac{243}{35}\approx 6.943.$$

Hence the center of mass is
Equation:

$$\bar{x}=\frac{M_{yz}}{m},\bar{y}=\frac{M_{xz}}{m},\bar{z}=\frac{M_{xy}}{m},$$
$$\bar{x}=\frac{M_{yz}}{m}=\frac{243/35}{108/35}=\frac{243}{108}=2.25,$$
$$\bar{y}=\frac{M_{xz}}{m}=\frac{81/35}{108/35}=\frac{81}{108}=0.75,$$
$$\bar{z}=\frac{M_{xy}}{m}=\frac{54/35}{108/35}=\frac{54}{108}=0.5.$$

The center of mass for the tetrahedron Q is the point $(2.25,0.75,0.5)$.

Note:

Exercise:

Problem:

Consider the same region Q ([link]) and use the density function $\rho\left(x,y,z\right)=xy^2z$. Find the center of mass.

Solution:

$\left(\frac{3}{2},\frac{9}{8},\frac{1}{2}\right)$

Hint

Check that $M_{xy}=\frac{27}{35}$, $M_{xz}=\frac{243}{140}$, and $M_{yz}=\frac{81}{35}$. Then use m from a previous checkpoint question.

We conclude this section with an example of finding moments of inertia I_x, I_y, and I_z.

Example:

Exercise:

Problem:

Finding the Moments of Inertia of a Solid

Suppose that Q is a solid region and is bounded by $x+2y+3z=6$ and the coordinate planes with density $\rho\left(x,y,z\right)=x^2yz$ (see [link]). Find the moments of inertia of the tetrahedron Q about the yz-plane, the xz-plane, and the xy-plane.

Solution:

Once again, we can almost immediately write the limits of integration and hence we can quickly proceed to evaluating the moments of inertia. Using the formula stated before, the moments of inertia of the tetrahedron Q about the xy-plane, the xz-plane, and the yz-plane are

Equation:

$$I_x=\iiint\limits_Q\left(y^2+z^2\right)\rho\left(x,y,z\right)dV,$$
$$I_y=\iiint\limits_Q\left(x^2+z^2\right)\rho\left(x,y,z\right)dV,$$

and

Equation:

$$I_z=\iiint\limits_Q\left(x^2+y^2\right)\rho\left(x,y,z\right)dV \text{ with } \rho\left(x,y,z\right)=x^2yz.$$

Proceeding with the computations, we have

Equation:

$$I_x = \iiint\limits_Q (y^2+z^2)x^2yz\,dV = \int\limits_{x=0}^{x=6}\int\limits_{y=0}^{y=\frac{1}{2}(6-x)}\int\limits_{z=0}^{z=\frac{1}{3}(6-x-2y)} (y^2+z^2)x^2yz\,dz\,dy\,dx = \frac{117}{35} \approx 3.343,$$

$$I_y = \iiint\limits_Q (x^2+z^2)x^2yz\,dV = \int\limits_{x=0}^{x=6}\int\limits_{y=0}^{y=\frac{1}{2}(6-x)}\int\limits_{z=0}^{z=\frac{1}{3}(6-x-2y)} (x^2+z^2)x^2yz\,dz\,dy\,dx = \frac{684}{35} \approx 19.543,$$

$$I_z = \iiint\limits_Q (x^2+y^2)x^2yz\,dV = \int\limits_{x=0}^{x=6}\int\limits_{y=0}^{y=\frac{1}{2}(6-x)}\int\limits_{z=0}^{z=\frac{1}{3}(6-x-2y)} (x^2+y^2)x^2yz\,dz\,dy\,dx = \frac{729}{35} \approx 20.829.$$

Thus, the moments of inertia of the tetrahedron Q about the yz-plane, the xz-plane, and the xy-plane are $117/35, 684/35$, and $729/35$, respectively.

Note:
Exercise:

Problem:

Consider the same region Q ([link]), and use the density function $\rho(x,y,z) = xy^2z$. Find the moments of inertia about the three coordinate planes.

Solution:

The moments of inertia of the tetrahedron Q about the yz-plane, the xz-plane, and the xy-plane are $99/35, 36/7$, and $243/35$, respectively.

Key Concepts

Finding the mass, center of mass, moments, and moments of inertia in double integrals:

- For a lamina R with a density function $\rho(x,y)$ at any point (x,y) in the plane, the mass is $m = \iint\limits_R \rho(x,y)dA$.
- The moments about the x-axis and y-axis are
  Equation:

$$M_x = \iint\limits_R y\rho(x,y)dA \text{ and } M_y = \iint\limits_R x\rho(x,y)dA.$$

- The center of mass is given by $\bar{x} = \frac{M_y}{m}, \bar{y} = \frac{M_x}{m}$.
- The center of mass becomes the centroid of the plane when the density is constant.
- The moments of inertia about the $x-$axis, $y-$axis, and the origin are

Equation:

$$I_x = \iint\limits_R y^2 \rho(x,y)dA, \;\; I_y = \iint\limits_R x^2 \rho(x,y)dA, \text{ and } I_0 = I_x + I_y = \iint\limits_R \left(x^2 + y^2\right) \rho(x,y)dA.$$

Finding the mass, center of mass, moments, and moments of inertia in triple integrals:

- For a solid object Q with a density function $\rho(x,y,z)$ at any point (x,y,z) in space, the mass is $m = \iiint\limits_Q \rho(x,y,z)dV.$
- The moments about the xy-plane, the xz-plane, and the yz-plane are
  Equation:

$$M_{xy} = \iiint\limits_Q z\rho(x,y,z)dV, \; M_{xz} = \iiint\limits_Q y\rho(x,y,z)dV, \; M_{yz} = \iiint\limits_Q x\rho(x,y,z)dV.$$

- The center of mass is given by $\bar{x} = \frac{M_{yz}}{m}, \bar{y} = \frac{M_{xz}}{m}, \bar{z} = \frac{M_{xy}}{m}.$
- The center of mass becomes the centroid of the solid when the density is constant.
- The moments of inertia about the yz-plane, the xz-plane, and the xy-plane are
  Equation:

$$I_x = \iiint\limits_Q \left(y^2 + z^2\right)\rho(x,y,z)dV, I_y = \iiint\limits_Q \left(x^2 + z^2\right)\rho(x,y,z)dV,$$
$$I_z = \iiint\limits_Q \left(x^2 + y^2\right)\rho(x,y,z)dV.$$

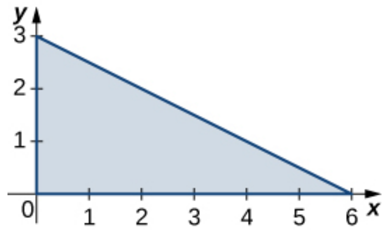
Key Equations

- **Mass of a lamina**
  $m = \lim\limits_{k,l\to\infty} \sum\limits_{i=1}^{k}\sum\limits_{j=1}^{l} m_{ij} = \lim\limits_{k,l\to\infty} \sum\limits_{i=1}^{k}\sum\limits_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint\limits_R \rho(x,y)dA$
- **Moment about the *x*-axis**
  $M_x = \lim\limits_{k,l\to\infty} \sum\limits_{i=1}^{k}\sum\limits_{j=1}^{l} \left(y_{ij}^*\right) m_{ij} = \lim\limits_{k,l\to\infty} \sum\limits_{i=1}^{k}\sum\limits_{j=1}^{l} \left(y_{ij}^*\right) \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint\limits_R y\rho(x,y)dA$
- **Moment about the *y*-axis**
  $M_y = \lim\limits_{k,l\to\infty} \sum\limits_{i=1}^{k}\sum\limits_{j=1}^{l} \left(x_{ij}^*\right) m_{ij} = \lim\limits_{k,l\to\infty} \sum\limits_{i=1}^{k}\sum\limits_{j=1}^{l} \left(x_{ij}^*\right) \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint\limits_R x\rho(x,y)dA$
- **Center of mass of a lamina**
  $\bar{x} = \frac{M_y}{m} = \frac{\iint\limits_R x\rho(x,y)dA}{\iint\limits_R \rho(x,y)dA}$ and $\bar{y} = \frac{M_x}{m} = \frac{\iint\limits_R y\rho(x,y)dA}{\iint\limits_R \rho(x,y)dA}$

In the following exercises, the region R occupied by a lamina is shown in a graph. Find the mass of R with the density function ρ.

Exercise:

Problem: R is the triangular region with vertices $(0,0)$, $(0,3)$, and $(6,0)$; $\rho(x,y) = xy$.

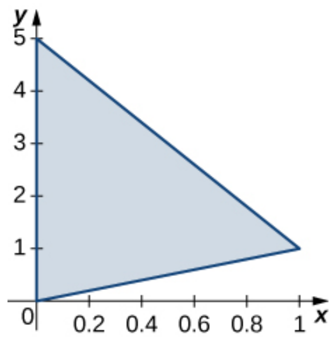


Solution:

$\frac{27}{2}$

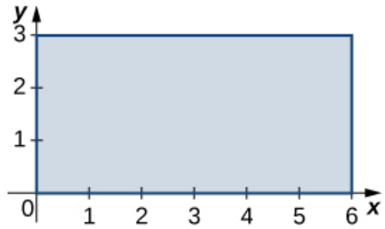
Exercise:

Problem: R is the triangular region with vertices $(0,0)$, $(1,1)$, $(0,5)$; $\rho(x,y) = x + y$.



Exercise:

Problem: R is the rectangular region with vertices $(0,0)$, $(0,3)$, $(6,3)$, and $(6,0)$; $\rho(x,y) = \sqrt{xy}$.

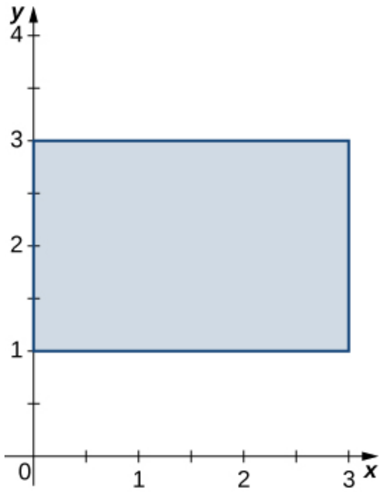


Solution:

$24\sqrt{2}$

Exercise:

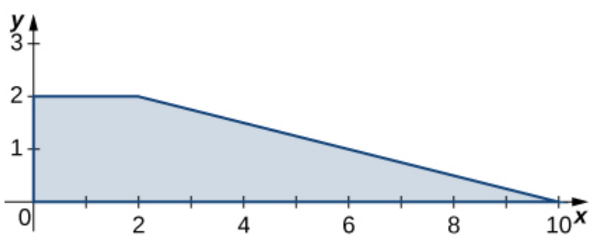
Problem: R is the rectangular region with vertices $(0, 1)$, $(0, 3)$, $(3, 3)$, and $(3, 1)$; $\rho(x, y) = x^2 y$.



Exercise:

Problem:

R is the trapezoidal region determined by the lines $y = -\frac{1}{4}x + \frac{5}{2}$, $y = 0$, $y = 2$, and $x = 0$; $\rho(x, y) = 3xy$.



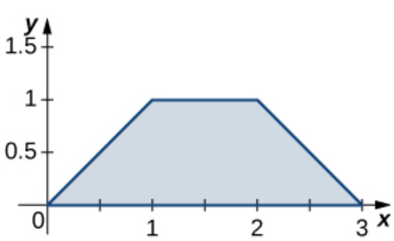
Solution:

76

Exercise:

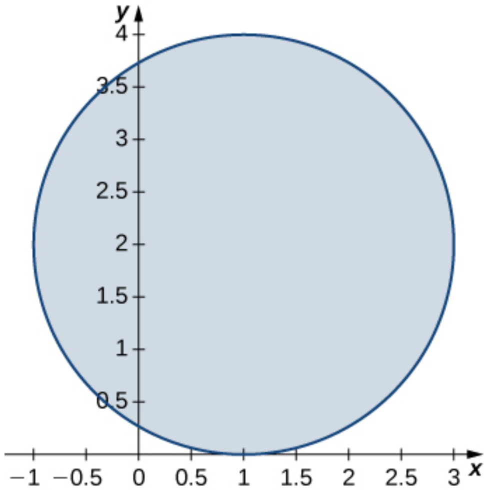
Problem:

R is the trapezoidal region determined by the lines $y = 0$, $y = 1$, $y = x$, and $y = -x + 3$; $\rho(x, y) = 2x + y$.



Exercise:

Problem: R is the disk of radius 2 centered at $(1,2)$; $\rho(x,y)=x^2+y^2-2x-4y+5$.

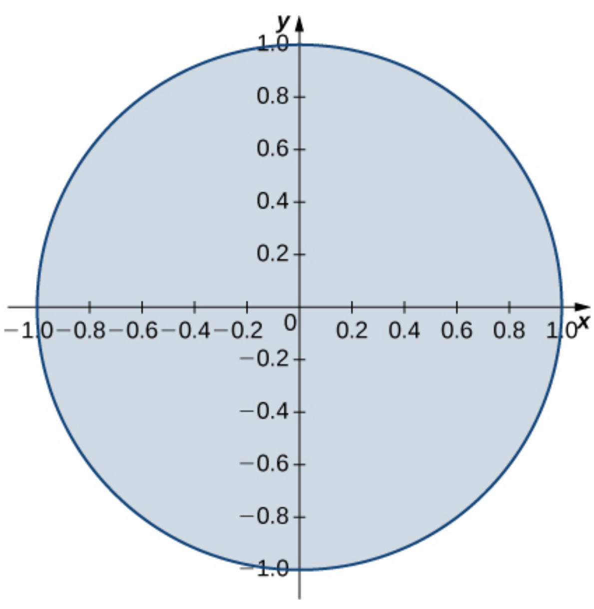


Solution:

8π

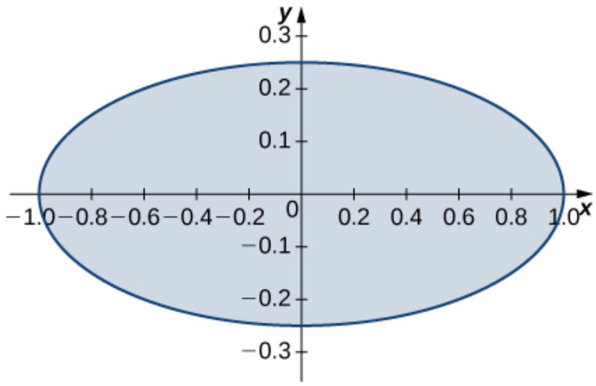
Exercise:

Problem: R is the unit disk; $\rho(x,y)=3x^4+6x^2y^2+3y^4$.



Exercise:

Problem: R is the region enclosed by the ellipse $x^2+4y^2=1$; $\rho(x,y)=1$.

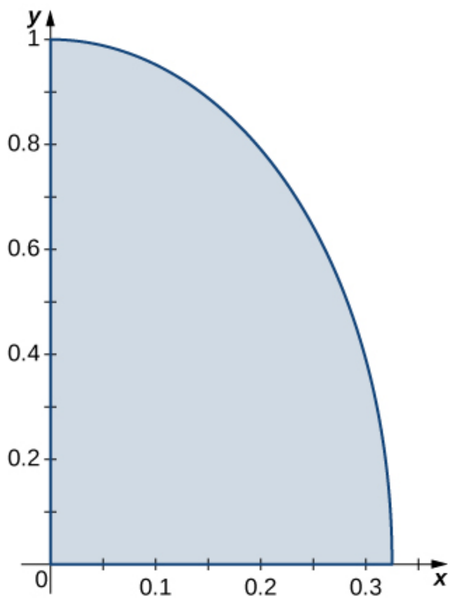


Solution:

$\frac{\pi}{2}$

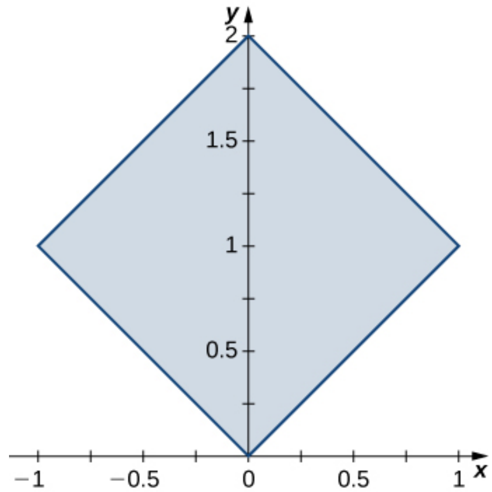
Exercise:

Problem: $R = \{(x, y) | 9x^2 + y^2 \leq 1, x \geq 0, y \geq 0\}; \rho(x, y) = \sqrt{9x^2 + y^2}.$



Exercise:

Problem: R is the region bounded by $y = x, y = -x, y = x + 2, y = -x + 2; \rho(x, y) = 1.$

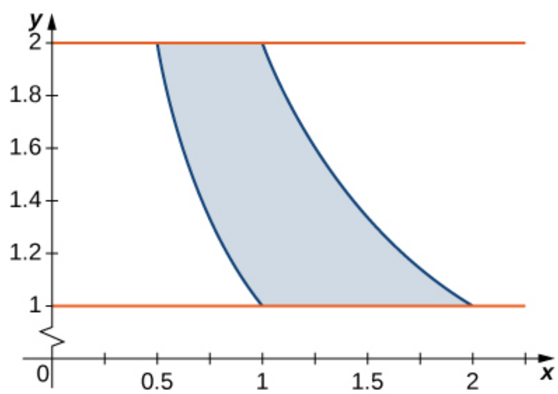


Solution:

2

Exercise:

Problem: R is the region bounded by $y = \frac{1}{x}, y = \frac{2}{x}, y = 1$, and $y = 2$; $\rho(x, y) = 4(x + y)$.



In the following exercises, consider a lamina occupying the region R and having the density function ρ given in the preceding group of exercises. Use a computer algebra system (CAS) to answer the following questions.

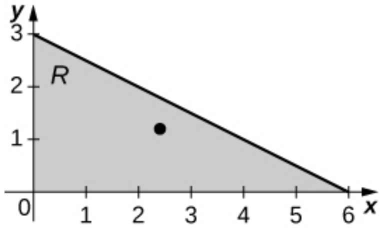
a. Find the moments M_x and M_y about the x-axis and y-axis, respectively.
b. Calculate and plot the center of mass of the lamina.
c. **[T]** Use a CAS to locate the center of mass on the graph of R.

Exercise:

Problem: **[T]** R is the triangular region with vertices $(0, 0)$, $(0, 3)$, and $(6, 0)$; $\rho(x, y) = xy$.

Solution:

a. $M_x = \frac{81}{5}, M_y = \frac{162}{5}$; b. $\bar{x} = \frac{12}{5}, \bar{y} = \frac{6}{5}$;
c.



Exercise:

Problem: **[T]** R is the triangular region with vertices $(0,0), (1,1),$ and $(0,5)$; $\rho(x,y) = x + y$.

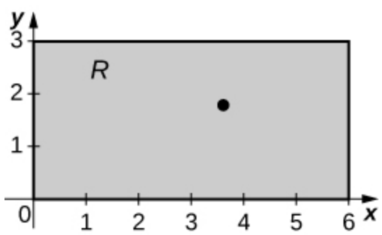
Exercise:

Problem:

[T] R is the rectangular region with vertices $(0,0), (0,3), (6,3),$ and $(6,0)$; $\rho(x,y) = \sqrt{xy}$.

Solution:

a. $M_x = \frac{216\sqrt{2}}{5}, M_y = \frac{432\sqrt{2}}{5}$; b. $\bar{x} = \frac{18}{5}, \bar{y} = \frac{9}{5}$;
c.



Exercise:

Problem: **[T]** R is the rectangular region with vertices $(0,1), (0,3), (3,3),$ and $(3,1)$; $\rho(x,y) = x^2 y$.

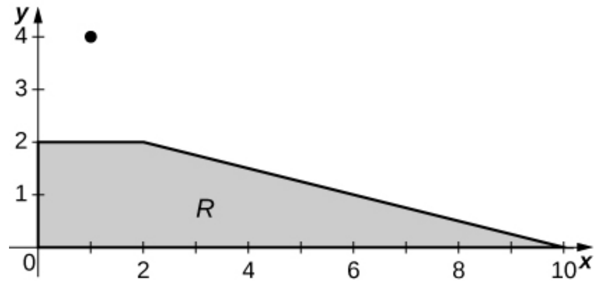
Exercise:

Problem:

[T] R is the trapezoidal region determined by the lines $y = -\frac{1}{4}x + \frac{5}{2}, y = 0, y = 2,$ and $x = 0$; $\rho(x,y) = 3xy$.

Solution:

a. $M_x = \frac{368}{5}, M_y = \frac{1552}{5}$; b. $\bar{x} = \frac{92}{95}, \bar{y} = \frac{388}{95}$;
c.



Exercise:

Problem:

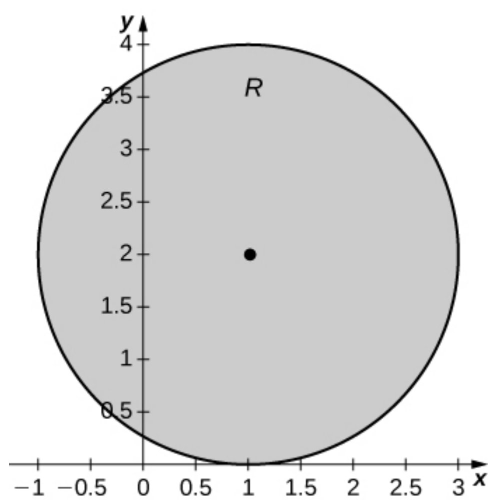
[T] R is the trapezoidal region determined by the lines $y = 0, y = 1, y = x$, and $y = -x + 3; \rho(x, y) = 2x + y$.

Exercise:

Problem: **[T]** R is the disk of radius 2 centered at $(1, 2); \rho(x, y) = x^2 + y^2 - 2x - 4y + 5$.

Solution:

a. $M_x = 16\pi, M_y = 8\pi$; b. $\bar{x} = 1, \bar{y} = 2$;
c.



Exercise:

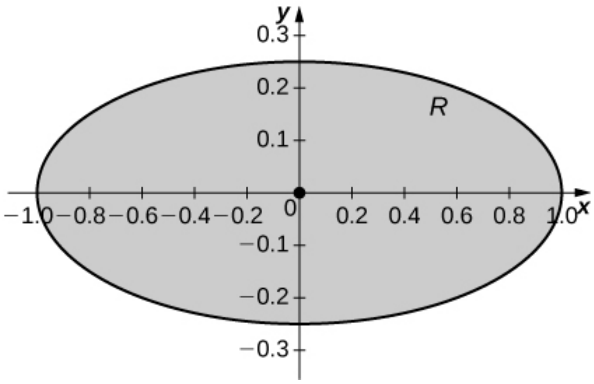
Problem: **[T]** R is the unit disk; $\rho(x, y) = 3x^4 + 6x^2y^2 + 3y^4$.

Exercise:

Problem: **[T]** R is the region enclosed by the ellipse $x^2 + 4y^2 = 1; \rho(x, y) = 1$.

Solution:

a. $M_x = 0, M_y = 0$; b. $\bar{x} = 0, \bar{y} = 0$;
c.



Exercise:

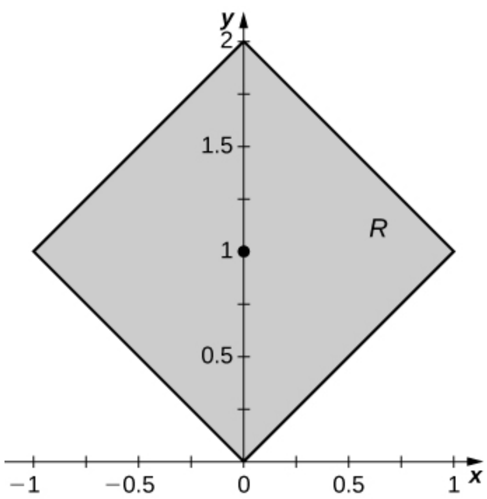
Problem: **[T]** $R = \left\{(x,y) | 9x^2 + y^2 \leq 1, x \geq 0, y \geq 0\right\}$; $\rho(x,y) = \sqrt{9x^2 + y^2}$.

Exercise:

Problem: **[T]** R is the region bounded by $y = x, y = -x, y = x + 2$, and $y = -x + 2$; $\rho(x,y) = 1$.

Solution:

a. $M_x = 2, M_y = 0$; b. $\bar{x} = 0, \bar{y} = 1$;
c.



Exercise:

Problem: **[T]** R is the region bounded by $y = \frac{1}{x}, y = \frac{2}{x}, y = 1$, and $y = 2$; $\rho(x,y) = 4(x + y)$.

In the following exercises, consider a lamina occupying the region R and having the density function ρ given in the first two groups of Exercises.

a. Find the moments of inertia I_x, I_y, and I_0 about the x-axis, y-axis, and origin, respectively.
b. Find the radii of gyration with respect to the x-axis, y-axis, and origin, respectively.

Exercise:

Problem: R is the triangular region with vertices $(0,0)$, $(0,3)$, and $(6,0)$; $\rho(x,y)=xy$.

Solution:

a. $I_x=\frac{243}{10}$, $I_y=\frac{486}{5}$, and $I_0=\frac{243}{2}$; b. $R_x=\frac{3\sqrt{5}}{5}$, $R_y=\frac{6\sqrt{5}}{5}$, and $R_0=3$

Exercise:

Problem: R is the triangular region with vertices $(0,0)$, $(1,1)$, and $(0,5)$; $\rho(x,y)=x+y$.

Exercise:

Problem: R is the rectangular region with vertices $(0,0)$, $(0,3)$, $(6,3)$, and $(6,0)$; $\rho(x,y)=\sqrt{xy}$.

Solution:

a. $I_x=\frac{2592\sqrt{2}}{7}$, $I_y=\frac{648\sqrt{2}}{7}$, and $I_0=\frac{3240\sqrt{2}}{7}$; b. $R_x=\frac{6\sqrt{21}}{7}$, $R_y=\frac{3\sqrt{21}}{7}$, and $R_0=\frac{3\sqrt{105}}{7}$

Exercise:

Problem: R is the rectangular region with vertices $(0,1)$, $(0,3)$, $(3,3)$, and $(3,1)$; $\rho(x,y)=x^2y$.

Exercise:

Problem:

R is the trapezoidal region determined by the lines $y=-\frac{1}{4}x+\frac{5}{2}$, $y=0$, $y=2$, and $x=0$; $\rho(x,y)=3xy$.

Solution:

a. $I_x=88$, $I_y=1560$, and $I_0=1648$; b. $R_x=\frac{\sqrt{418}}{19}$, $R_y=\frac{\sqrt{7410}}{19}$, and $R_0=\frac{2\sqrt{1957}}{19}$

Exercise:

Problem:

R is the trapezoidal region determined by the lines $y=0$, $y=1$, $y=x$, and $y=-x+3$; $\rho(x,y)=2x+y$.

Exercise:

Problem: R is the disk of radius 2 centered at $(1,2)$; $\rho(x,y)=x^2+y^2-2x-4y+5$.

Solution:

a. $I_x=\frac{128\pi}{3}$, $I_y=\frac{56\pi}{3}$, and $I_0=\frac{184\pi}{3}$; b. $R_x=\frac{4\sqrt{3}}{3}$, $R_y=\frac{\sqrt{21}}{3}$, and $R_0=\frac{\sqrt{69}}{3}$

Exercise:

Problem: R is the unit disk; $\rho(x, y) = 3x^4 + 6x^2y^2 + 3y^4$.

Exercise:

Problem: R is the region enclosed by the ellipse $x^2 + 4y^2 = 1$; $\rho(x, y) = 1$.

Solution:

a. $I_x = \frac{\pi}{32}, I_y = \frac{\pi}{8}$, and $I_0 = \frac{5\pi}{32}$; b. $R_x = \frac{1}{4}, R_y = \frac{1}{2}$, and $R_0 = \frac{\sqrt{5}}{4}$

Exercise:

Problem: $R = \left\{(x, y) | 9x^2 + y^2 \leq 1, x \geq 0, y \geq 0\right\}; \rho(x, y) = \sqrt{9x^2 + y^2}$.

Exercise:

Problem: R is the region bounded by $y = x, y = -x, y = x + 2$, and $y = -x + 2$; $\rho(x, y) = 1$.

Solution:

a. $I_x = \frac{7}{3}, I_y = \frac{1}{3}$, and $I_0 = \frac{8}{3}$; b. $R_x = \frac{\sqrt{42}}{6}, R_y = \frac{\sqrt{6}}{6}$, and $R_0 = \frac{2\sqrt{3}}{3}$

Exercise:

Problem: R is the region bounded by $y = \frac{1}{x}, y = \frac{2}{x}, y = 1$, and $y = 2$; $\rho(x, y) = 4(x + y)$.

Exercise:

Problem:

Let Q be the solid unit cube. Find the mass of the solid if its density ρ is equal to the square of the distance of an arbitrary point of Q to the xy-plane.

Solution:

$m = \frac{1}{3}$

Exercise:

Problem:

Let Q be the solid unit hemisphere. Find the mass of the solid if its density ρ is proportional to the distance of an arbitrary point of Q to the origin.

Exercise:

Problem:

The solid Q of constant density 1 is situated inside the sphere $x^2 + y^2 + z^2 = 16$ and outside the sphere $x^2 + y^2 + z^2 = 1$. Show that the center of mass of the solid is not located within the solid.

Exercise:

Problem:

Find the mass of the solid $Q = \left\{(x, y, z)|1 \le x^2 + z^2 \le 25, y \le 1 - x^2 - z^2\right\}$ whose density is $\rho(x, y, z) = k$, where $k > 0$.

Exercise:

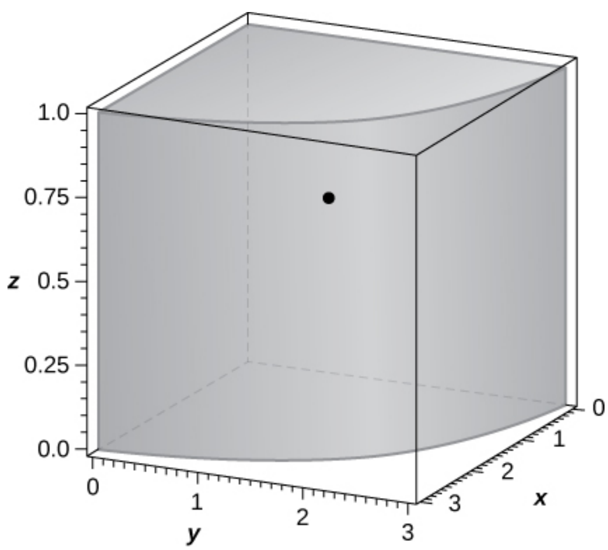
Problem:

[T] The solid $Q = \left\{(x, y, z)|x^2 + y^2 \le 9, 0 \le z \le 1, x \ge 0, y \ge 0\right\}$ has density equal to the distance to the xy-plane. Use a CAS to answer the following questions.

a. Find the mass of Q.
b. Find the moments $M_{xy}, M_{xz},$ and M_{yz} about the xy-plane, xz-plane, and yz-plane, respectively.
c. Find the center of mass of Q.
d. Graph Q and locate its center of mass.

Solution:

a. $m = \frac{9\pi}{4}$; b. $M_{xy} = \frac{3\pi}{2}, M_{xz} = \frac{81}{8}, M_{yz} = \frac{81}{8}$; c. $\bar{x} = \frac{9}{2\pi}, \bar{y} = \frac{9}{2\pi}, \bar{z} = \frac{2}{3}$; d. the solid Q and its center of mass are shown in the following figure.



Exercise:

Problem:

Consider the solid $Q = \{(x, y, z)|0 \le x \le 1, 0 \le y \le 2, 0 \le z \le 3\}$ with the density function $\rho(x, y, z) = x + y + 1$.

a. Find the mass of Q.
b. Find the moments $M_{xy}, M_{xz},$ and M_{yz} about the xy-plane, xz-plane, and yz-plane, respectively.
c. Find the center of mass of Q.

Exercise:

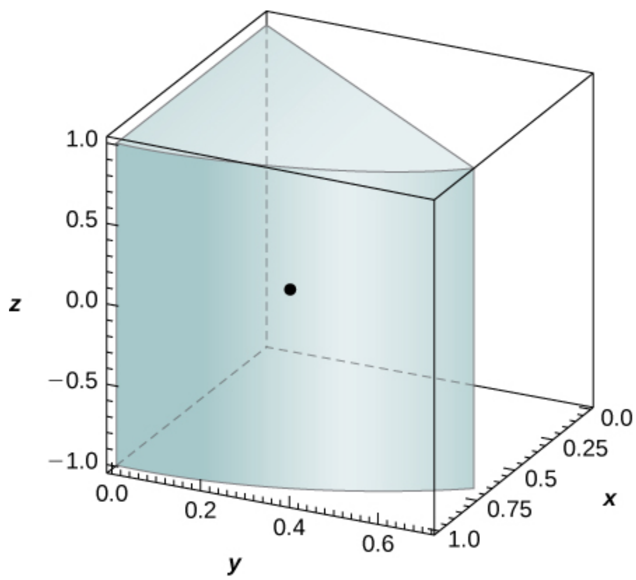
Problem:

[T] The solid Q has the mass given by the triple integral $\int_{-1}^{1}\int_{0}^{\frac{\pi}{4}}\int_{0}^{1} r^2 dr\, d\theta\, dz$. Use a CAS to answer the following questions.

a. Show that the center of mass of Q is located in the xy-plane.
b. Graph Q and locate its center of mass.

Solution:

a. $\bar{x} = \frac{3\sqrt{2}}{2\pi}, \bar{y} = \frac{3\left(2-\sqrt{2}\right)}{2\pi}, \bar{z} = 0$; b. the solid Q and its center of mass are shown in the following figure.



Exercise:

Problem:

The solid Q is bounded by the planes $x + 4y + z = 8, x = 0, y = 0,$ and $z = 0$. Its density at any point is equal to the distance to the xz-plane. Find the moments of inertia I_y of the solid about the xz-plane.

Exercise:

Problem:

The solid Q is bounded by the planes $x + y + z = 3$, $x = 0, y = 0$, and $z = 0$. Its density is $\rho\left(x, y, z\right) = x + ay$, where $a > 0$. Show that the center of mass of the solid is located in the plane $z = \frac{3}{5}$ for any value of a.

Exercise:

Problem:

Let Q be the solid situated outside the sphere $x^2+y^2+z^2=z$ and inside the upper hemisphere $x^2+y^2+z^2=R^2$, where $R>1$. If the density of the solid is $\rho(x,y,z)=\frac{1}{\sqrt{x^2+y^2+z^2}}$, find R such that the mass of the solid is $\frac{7\pi}{2}$.

Exercise:

Problem:

The mass of a solid Q is given by $\int\limits_0^2 \int\limits_0^{\sqrt{4-x^2}} \int\limits_{\sqrt{x^2+y^2}}^{\sqrt{16-x^2-y^2}} \left(x^2+y^2+z^2\right)^n dz\,dy\,dx$, where n is an integer. Determine n such the mass of the solid is $\left(2-\sqrt{2}\right)\pi$.

Solution:

$n=-2$

Exercise:

Problem:

Let Q be the solid bounded above the cone $x^2+y^2=z^2$ and below the sphere $x^2+y^2+z^2-4z=0$. Its density is a constant $k>0$. Find k such that the center of mass of the solid is situated 7 units from the origin.

Exercise:

Problem:

The solid $Q=\left\{(x,y,z)|0\le x^2+y^2\le 16, x\ge 0, y\ge 0, 0\le z\le x\right\}$ has the density $\rho(x,y,z)=k$. Show that the moment M_{xy} about the xy-plane is half of the moment M_{yz} about the yz-plane.

Exercise:

Problem:

The solid Q is bounded by the cylinder $x^2+y^2=a^2$, the paraboloid $b^2-z=x^2+y^2$, and the xy-plane, where $0<a<b$. Find the mass of the solid if its density is given by $\rho(x,y,z)=\sqrt{x^2+y^2}$.

Exercise:

Problem:

Let Q be a solid of constant density k, where $k>0$, that is located in the first octant, inside the circular cone $x^2+y^2=9(z-1)^2$, and above the plane $z=0$. Show that the moment M_{xy} about the xy-plane is the same as the moment M_{yz} about the xz-plane.

Exercise:

Problem: The solid Q has the mass given by the triple integral $\int\limits_0^1 \int\limits_0^{\pi/2} \int\limits_0^{r^2} \left(r^4+r\right) dz\,d\theta\,dr$.

a. Find the density of the solid in rectangular coordinates.
b. Find the moment M_{xy} about the xy-plane.

Exercise:

Problem:

The solid Q has the moment of inertia I_x about the yz-plane given by the triple integral $\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{\frac{1}{2}(x^2+y^2)}^{\sqrt{x^2+y^2}} \left(y^2+z^2\right)\left(x^2+y^2\right) dz\, dx\, dy.$

a. Find the density of Q.
b. Find the moment of inertia I_z about the xy-plane.

Solution:

a. $\rho\left(x,y,z\right)=x^2+y^2$; b. $\frac{16\pi}{7}$

Exercise:

Problem:

The solid Q has the mass given by the triple integral $\int_0^{\pi/4} \int_0^{2\sec\theta} \int_0^1 \left(r^3\cos\theta\sin\theta+2r\right) dz\, dr\, d\theta.$

a. Find the density of the solid in rectangular coordinates.
b. Find the moment M_{xz} about the xz-plane.

Exercise:

Problem:

Let Q be the solid bounded by the xy-plane, the cylinder $x^2+y^2=a^2$, and the plane $z=1$, where $a>1$ is a real number. Find the moment M_{xy} of the solid about the xy-plane if its density given in cylindrical coordinates is $\rho\left(r,\theta,z\right)=\frac{d^2 f}{dr^2}(r)$, where f is a differentiable function with the first and second derivatives continuous and differentiable on $(0,a)$.

Solution:

$M_{xy}=\pi\left(f\left(0\right)-f\left(a\right)+af'\left(a\right)\right)$

Exercise:

Problem:

A solid Q has a volume given by $\iint_D \int_a^b dA\, dz$, where D is the projection of the solid onto the xy-plane and $a<b$ are real numbers, and its density does not depend on the variable z. Show that its center of mass lies in the plane $z=\frac{a+b}{2}$.

Exercise:

Problem:

Consider the solid enclosed by the cylinder $x^2 + z^2 = a^2$ and the planes $y = b$ and $y = c$, where $a > 0$ and $b < c$ are real numbers. The density of Q is given by $\rho(x, y, z) = f\prime(y)$, where f is a differential function whose derivative is continuous on (b, c). Show that if $f(b) = f(c)$, then the moment of inertia about the xz-plane of Q is null.

Exercise:

Problem:

[T] The average density of a solid Q is defined as $\rho_{ave} = \frac{1}{V(Q)} \iiint\limits_Q \rho(x, y, z) dV = \frac{m}{V(Q)}$, where $V(Q)$ and m are the volume and the mass of Q, respectively. If the density of the unit ball centered at the origin is $\rho(x, y, z) = e^{-x^2-y^2-z^2}$, use a CAS to find its average density. Round your answer to three decimal places.

Exercise:

Problem:

Show that the moments of inertia I_x, I_y, and I_z about the yz-plane, xz-plane, and xy-plane, respectively, of the unit ball centered at the origin whose density is $\rho(x, y, z) = e^{-x^2-y^2-z^2}$ are the same. Round your answer to two decimal places.

Solution:

$I_x = I_y = I_z \simeq 0.84$

Glossary

radius of gyration
: the distance from an object’s center of mass to its axis of rotation

Change of Variables in Multiple Integrals

- Determine the image of a region under a given transformation of variables.
- Compute the Jacobian of a given transformation.
- Evaluate a double integral using a change of variables.
- Evaluate a triple integral using a change of variables.

Recall from [Substitution Rule] the method of integration by substitution. When evaluating an integral such as $\int_2^3 x(x^2-4)^5 dx$, we substitute $u = g(x) = x^2 - 4$. Then $du = 2x\, dx$ or $x\, dx = \frac{1}{2} du$ and the limits change to $u = g(2) = 2^2 - 4 = 0$ and $u = g(3) = 9 - 4 = 5$. Thus the integral becomes $\int_0^5 \frac{1}{2} u^5 du$ and this integral is much simpler to evaluate. In other words, when solving integration problems, we make appropriate substitutions to obtain an integral that becomes much simpler than the original integral.

We also used this idea when we transformed double integrals in rectangular coordinates to polar coordinates and transformed triple integrals in rectangular coordinates to cylindrical or spherical coordinates to make the computations simpler. More generally,
Equation:

$$\int\limits_a^b f(x)dx = \int\limits_c^d f(g(u))g\prime(u)du,$$

Where $x = g(u), dx = g\prime(u)du$, and $u = c$ and $u = d$ satisfy $c = g(a)$ and $d = g(b)$.

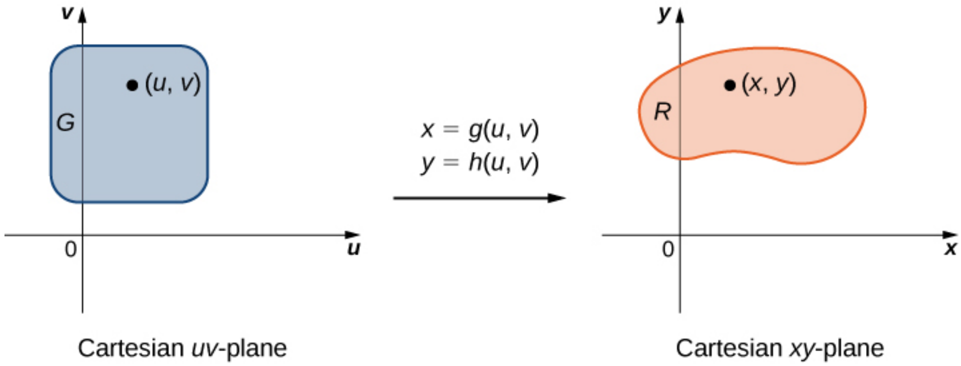
A similar result occurs in double integrals when we substitute $x = f(r, \theta) = r \cos \theta$, $y = g(r, \theta) = r \sin \theta$, and $dA = dx\, dy = r\, dr\, d\theta$. Then we get
Equation:

$$\iint\limits_R f(x, y)dA = \iint\limits_S f(r \cos \theta, r \sin \theta)r\, dr\, d\theta$$

where the domain R is replaced by the domain S in polar coordinates. Generally, the function that we use to change the variables to make the integration simpler is called a **transformation** or mapping.

Planar Transformations

A **planar transformation** T is a function that transforms a region G in one plane into a region R in another plane by a change of variables. Both G and R are subsets of R^2. For example, [link] shows a region G in the uv-plane transformed into a region R in the xy-plane by the change of variables $x = g(u, v)$ and $y = h(u, v)$, or sometimes we write $x = x(u, v)$ and $y = y(u, v)$. We shall typically assume that each of these functions has continuous first partial derivatives, which means g_u, g_v, h_u, and h_v exist and are also continuous. The need for this requirement will become clear soon.



The transformation of a region G in the uv-plane into a region R in the xy-plane.

Note:
Definition
A transformation $T\colon G \to R$, defined as $T\left(u,v\right)=\left(x,y\right)$, is said to be a **one-to-one transformation** if no two points map to the same image point.

To show that T is a one-to-one transformation, we assume $T\left(u_1,v_1\right)=T\left(u_2,v_2\right)$ and show that as a consequence we obtain $\left(u_1,v_1\right)=\left(u_2,v_2\right)$. If the transformation T is one-to-one in the domain G, then the inverse T^{-1} exists with the domain R such that $T^{-1}\circ T$ and $T\circ T^{-1}$ are identity functions.

[link] shows the mapping $T\left(u,v\right)=\left(x,y\right)$ where x and y are related to u and v by the equations $x=g\left(u,v\right)$ and $y=h\left(u,v\right)$. The region G is the domain of T and the region R is the range of T, also known as the *image* of G under the transformation T.

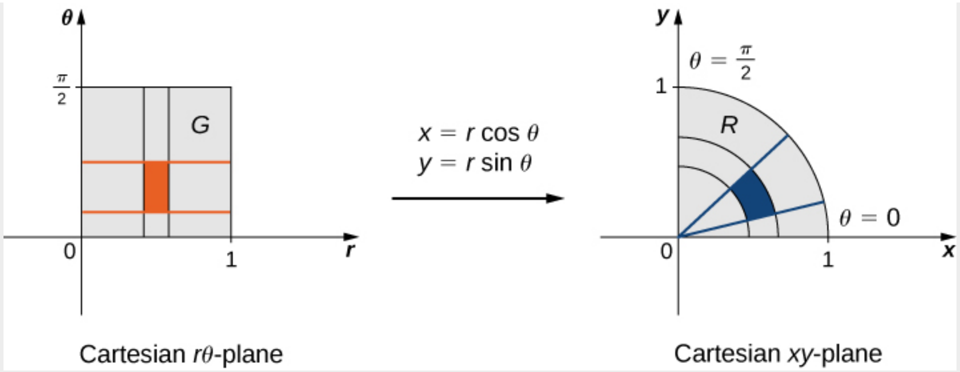
Example:
Exercise:

Problem:
Determining How the Transformation Works

Suppose a transformation T is defined as $T\left(r,\theta\right)=\left(x,y\right)$ where $x=r\cos\theta, y=r\sin\theta$. Find the image of the polar rectangle $G=\{(r,\theta)|0<r\le 1, 0\le\theta\le\pi/2\}$ in the $r\theta$-plane to a region R in the xy-plane. Show that T is a one-to-one transformation in G and find $T^{-1}\left(x,y\right)$.

Solution:

Since r varies from 0 to 1 in the $r\theta$-plane, we have a circular disc of radius 0 to 1 in the xy-plane. Because θ varies from 0 to $\pi/2$ in the $r\theta$-plane, we end up getting a quarter circle of radius 1 in the first quadrant of the xy-plane ([link]). Hence R is a quarter circle bounded by $x^2+y^2=1$ in the first quadrant.



A rectangle in the $r\theta$-plane is mapped into a quarter circle in the xy-plane.

In order to show that T is a one-to-one transformation, assume $T(r_1, \theta_1) = T(r_2, \theta_2)$ and show as a consequence that $(r_1, \theta_1) = (r_2, \theta_2)$. In this case, we have

Equation:

$$\begin{aligned} T(r_1, \theta_1) &= T(r_2, \theta_2), \\ (x_1, y_1) &= (x_1, y_1), \\ (r_1 \cos\theta_1, r_1 \sin\theta_1) &= (r_2 \cos\theta_2, r_2 \sin\theta_2), \\ r_1 \cos\theta_1 &= r_2 \cos\theta_2, r_1 \sin\theta_1 = r_2 \sin\theta_2. \end{aligned}$$

Dividing, we obtain

Equation:

$$\begin{aligned} \frac{r_1 \cos\theta_1}{r_1 \sin\theta_1} &= \frac{r_2 \cos\theta_2}{r_2 \sin\theta_2} \\ \frac{\cos\theta_1}{\sin\theta_1} &= \frac{\cos\theta_2}{\sin\theta_2} \\ \tan\theta_1 &= \tan\theta_2 \\ \theta_1 &= \theta_2 \end{aligned}$$

since the tangent function is one-one function in the interval $0 \le \theta \le \pi/2$. Also, since $0 < r \le 1$, we have $r_1 = r_2, \theta_1 = \theta_2$. Therefore, $(r_1, \theta_1) = (r_2, \theta_2)$ and T is a one-to-one transformation from G into R.

To find $T^{-1}(x, y)$ solve for r, θ in terms of x, y. We already know that $r^2 = x^2 + y^2$ and $\tan\theta = \frac{y}{x}$. Thus $T^{-1}(x, y) = (r, \theta)$ is defined as $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$.

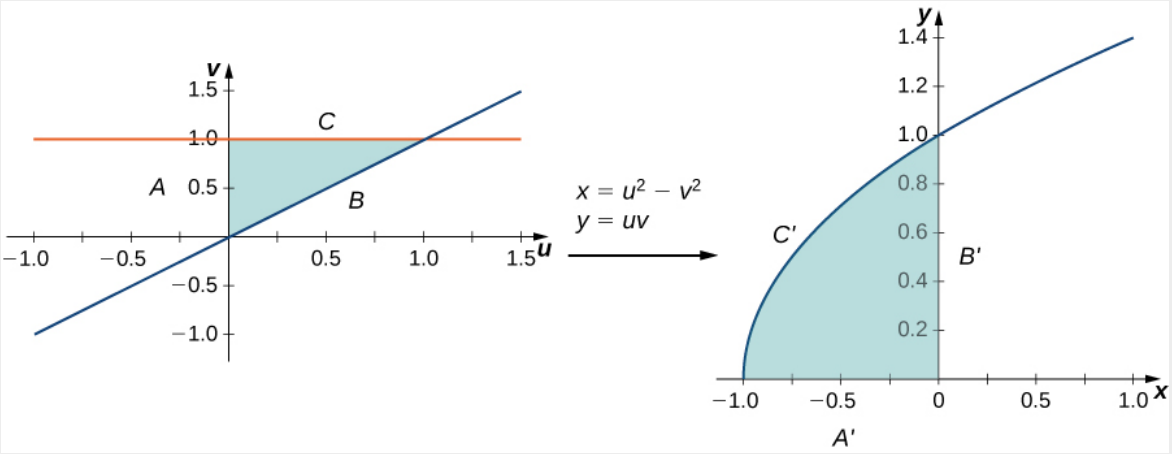
Example:
Exercise:

Problem:
Finding the Image under T

Let the transformation T be defined by $T(u, v) = (x, y)$ where $x = u^2 - v^2$ and $y = uv$. Find the image of the triangle in the uv-plane with vertices $(0, 0)$, $(0, 1)$, and $(1, 1)$.

Solution:

The triangle and its image are shown in [link]. To understand how the sides of the triangle transform, call the side that joins $(0,0)$ and $(0,1)$ side A, the side that joins $(0,0)$ and $(1,1)$ side B, and the side that joins $(1,1)$ and $(0,1)$ side C.



A triangular region in the uv-plane is transformed into an image in the xy-plane.

For the side A: $u=0, 0\le v\le 1$ transforms to $x=-v^2, y=0$ so this is the side $A\prime$ that joins $(-1,0)$ and $(0,0)$.

For the side B: $u=v, 0\le u\le 1$ transforms to $x=0, y=u^2$ so this is the side B' that joins $(0,0)$ and $(0,1)$.

For the side C: $0\le u\le 1, v=1$ transforms to $x=u^2-1, y=u$ (hence $x=y^2-1$) so this is the side C' that makes the upper half of the parabolic arc joining $(-1,0)$ and $(0,1)$.

All the points in the entire region of the triangle in the uv-plane are mapped inside the parabolic region in the xy-plane.

Note:
Exercise:

Problem:

Let a transformation T be defined as $T(u,v)=(x,y)$ where $x=u+v, y=3v$. Find the image of the rectangle $G=\{(u,v)\colon 0\le u\le 1, 0\le v\le 2\}$ from the uv-plane after the transformation into a region R in the xy-plane. Show that T is a one-to-one transformation and find $T^{-1}(x,y)$.

Solution:

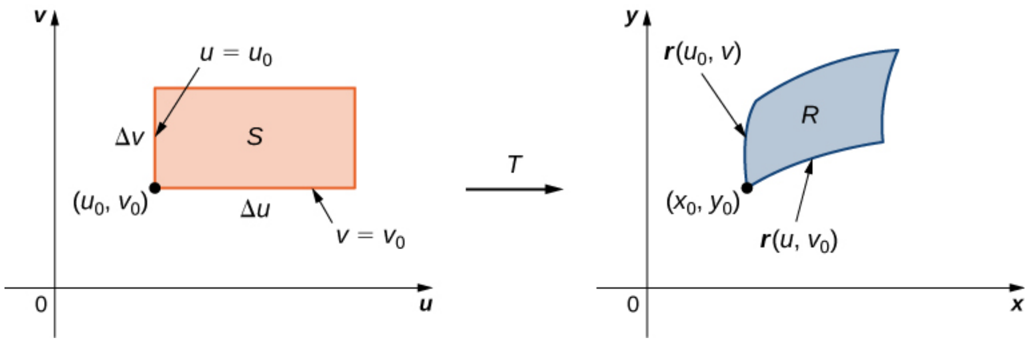
$T^{-1}(x,y)=(u,v)$ where $u=\frac{3x-y}{3}$ and $v=\frac{y}{3}$

Hint

Follow the steps of [link].

Jacobians

Recall that we mentioned near the beginning of this section that each of the component functions must have continuous first partial derivatives, which means that g_u, g_v, h_u, and h_v exist and are also continuous. A transformation that has this property is called a C^1 transformation (here C denotes continuous). Let $T(u, v) = (g(u, v), h(u, v))$, where $x = g(u, v)$ and $y = h(u, v)$, be a one-to-one C^1 transformation. We want to see how it transforms a small rectangular region S, Δu units by Δv units, in the uv-plane (see the following figure).



A small rectangle S in the uv-plane is transformed into a region R in the xy-plane.

Since $x = g(u, v)$ and $y = h(u, v)$, we have the position vector $\mathbf{r}(u, v) = g(u, v)\mathbf{i} + h(u, v)\mathbf{j}$ of the image of the point (u, v). Suppose that (u_0, v_0) is the coordinate of the point at the lower left corner that mapped to $(x_0, y_0) = T(u_0, v_0)$. The line $v = v_0$ maps to the image curve with vector function $\mathbf{r}(u, v_0)$, and the tangent vector at (x_0, y_0) to the image curve is

Equation:

$$\mathbf{r}_u = g_u(u_0, v_0)\mathbf{i} + h_u(u_0, v_0)\mathbf{j} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j}.$$

Similarly, the line $u = u_0$ maps to the image curve with vector function $\mathbf{r}(u_0, v)$, and the tangent vector at (x_0, y_0) to the image curve is

Equation:

$$\mathbf{r}_v = g_v(u_0, v_0)\mathbf{i} + h_v(u_0, v_0)\mathbf{j} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j}.$$

Now, note that

Equation:

$$\mathbf{r}_u = \lim_{\Delta u \to 0} \frac{\mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0)}{\Delta u} \text{ so } \mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0) \approx \Delta u \mathbf{r}_u.$$

Similarly,

Equation:

$$\mathbf{r}_v = \lim_{\Delta v \to 0} \frac{\mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0)}{\Delta v} \text{ so } \mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0) \approx \Delta v \mathbf{r}_v.$$

This allows us to estimate the area ΔA of the image R by finding the area of the parallelogram formed by the sides $\Delta v \mathbf{r}_v$ and $\Delta u \mathbf{r}_u$. By using the cross product of these two vectors by adding the **k**th component as 0, the area ΔA of the image R (refer to The Cross Product) is approximately $|\Delta u \mathbf{r}_u \times \Delta v \mathbf{r}_v| = |\mathbf{r}_u \times \mathbf{r}_v| \Delta u \Delta v$. In determinant form, the cross product is

Equation:

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \mathbf{k} = \left(\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right) \mathbf{k}.$$

Since $|\mathbf{k}| = 1$, we have $\Delta A \approx |\mathbf{r}_u \times \mathbf{r}_v| \Delta u \Delta v = \left(\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right) \Delta u \Delta v$.

Note:
Definition
The **Jacobian** of the C^1 transformation $T(u, v) = (g(u, v), h(u, v))$ is denoted by $J(u, v)$ and is defined by the 2×2 determinant

Equation:

$$J(u, v) = \left| \frac{\partial (x, y)}{\partial (u, v)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \left(\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right).$$

Using the definition, we have

Equation:

$$\Delta A \approx J(u, v) \Delta u \Delta v = \left| \frac{\partial (x, y)}{\partial (u, v)} \right| \Delta u \Delta v.$$

Note that the Jacobian is frequently denoted simply by

Equation:

$$J(u, v) = \frac{\partial (x, y)}{\partial (u, v)}.$$

Note also that

Equation:

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \left(\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

Hence the notation $J(u, v) = \frac{\partial (x, y)}{\partial (u, v)}$ suggests that we can write the Jacobian determinant with partials of x in the first row and partials of y in the second row.

Example:

Exercise:

Problem:
Finding the Jacobian

Find the Jacobian of the transformation given in [link].

Solution:

The transformation in the example is $T\left(r, \theta\right) = \left(r \cos \theta, r \sin \theta\right)$ where $x = r \cos \theta$ and $y = r \sin \theta$. Thus the Jacobian is

Equation:

$$\begin{aligned} J\left(r, \theta\right) &= \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} \\ &= r \cos^2\theta + r \sin^2\theta = r\left(\cos^2\theta + \sin^2\theta\right) = r. \end{aligned}$$

Example:
Exercise:

Problem:
Finding the Jacobian

Find the Jacobian of the transformation given in [link].

Solution:

The transformation in the example is $T\left(u, v\right) = \left(u^2 - v^2, uv\right)$ where $x = u^2 - v^2$ and $y = uv$. Thus the Jacobian is

Equation:

$$J\left(u, v\right) = \frac{\partial\left(x, y\right)}{\partial\left(u, v\right)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & v \\ -2v & u \end{vmatrix} = 2u^2 + 2v^2.$$

Note:
Exercise:

Problem: Find the Jacobian of the transformation given in the previous checkpoint: $T\left(u, v\right) = \left(u + v, 2v\right)$.

Solution:

$$J\left(u, v\right) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2$$

Hint

Follow the steps in the previous two examples.

Change of Variables for Double Integrals

We have already seen that, under the change of variables $T(u,v)=(x,y)$ where $x=g(u,v)$ and $y=h(u,v)$, a small region ΔA in the xy-plane is related to the area formed by the product $\Delta u\Delta v$ in the uv-plane by the approximation

Equation:

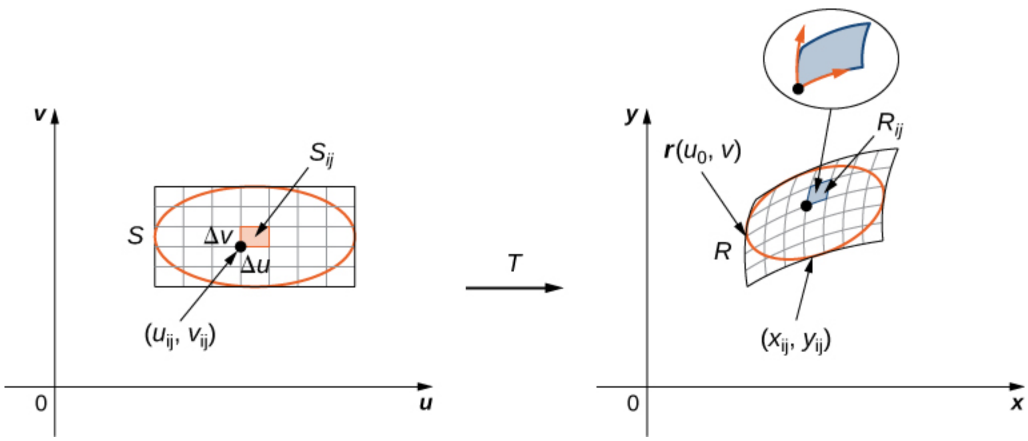
$$\Delta A\approx J(u,v)\Delta u,\Delta v.$$

Now let's go back to the definition of double integral for a minute:

Equation:

$$\iint\limits_R f(x,y)dA=\lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n}f(x_{ij},y_{ij})\Delta A.$$

Referring to [link], observe that we divided the region S in the uv-plane into small subrectangles S_{ij} and we let the subrectangles R_{ij} in the xy-plane be the images of S_{ij} under the transformation $T(u,v)=(x,y)$.



The subrectangles S_{ij} in the uv-plane transform into subrectangles R_{ij} in the xy-plane.

Then the double integral becomes

Equation:

$$\iint\limits_R f(x,y)dA=\lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n}f(x_{ij},y_{ij})\Delta A=\lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n}f(g(u_{ij},v_{ij}),h(u_{ij},v_{ij}))\,|J(u_{ij},v_{ij})|\Delta u\Delta v$$

Notice this is exactly the double Riemann sum for the integral

Equation:

$$\iint\limits_S f(g(u,v),h(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv.$$

Note:
Change of Variables for Double Integrals
Let $T(u,v)=(x,y)$ where $x=g(u,v)$ and $y=h(u,v)$ be a one-to-one C^1 transformation, with a nonzero Jacobian on the interior of the region S in the uv-plane; it maps S into the region R in the xy-plane. If f is continuous on R, then
Equation:

$$\iint_R f(x,y)dA=\iint_S f(g(u,v),h(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv.$$

With this theorem for double integrals, we can change the variables from (x,y) to (u,v) in a double integral simply by replacing
Equation:

$$dA=dx\,dy=\left|\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv$$

when we use the substitutions $x=g(u,v)$ and $y=h(u,v)$ and then change the limits of integration accordingly. This change of variables often makes any computations much simpler.

Example:
Exercise:

Problem:
Changing Variables from Rectangular to Polar Coordinates

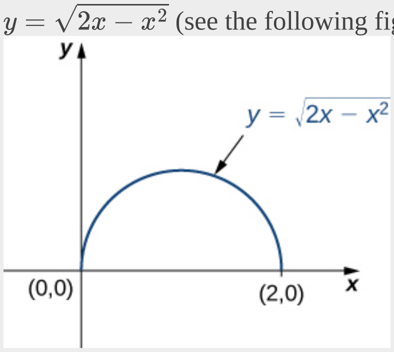
Consider the integral
Equation:

$$\int_0^2\int_0^{\sqrt{2x-x^2}}\sqrt{x^2+y^2}\,dy\,dx.$$

Use the change of variables $x=r\cos\theta$ and $y=r\sin\theta$, and find the resulting integral.

Solution:

First we need to find the region of integration. This region is bounded below by $y=0$ and above by $y=\sqrt{2x-x^2}$ (see the following figure).



Changing a region from rectangular to polar coordinates.

Squaring and collecting terms, we find that the region is the upper half of the circle $x^2+y^2-2x=0$, that is, $y^2+(x-1)^2=1$. In polar coordinates, the circle is $r=2\cos\theta$ so the region of integration in polar coordinates is bounded by $0\le r\le\cos\theta$ and $0\le\theta\le\frac{\pi}{2}$.

The Jacobian is $J(r,\theta)=r$, as shown in [link]. Since $r\ge 0$, we have $|J(r,\theta)|=r$.

The integrand $\sqrt{x^2+y^2}$ changes to r in polar coordinates, so the double iterated integral is

Equation:

$$\int_0^2\int_0^{\sqrt{2x-x^2}}\sqrt{x^2+y^2}\,dy\,dx=\int_0^{\pi/2}\int_0^{2\cos\theta} r\,|J(r,\theta)|\,dr\,d\theta=\int_0^{\pi/2}\int_0^{2\cos\theta} r^2\,dr\,d\theta.$$

Note:

Exercise:

Problem:

Considering the integral $\int_0^1\int_0^{\sqrt{1-x^2}}\left(x^2+y^2\right)dy\,dx$, use the change of variables $x=r\cos\theta$ and $y=r\sin\theta$, and find the resulting integral.

Solution:

$$\int_0^{\pi/2}\int_0^1 r^3\,dr\,d\theta$$

Hint

Follow the steps in the previous example.

Notice in the next example that the region over which we are to integrate may suggest a suitable transformation for the integration. This is a common and important situation.

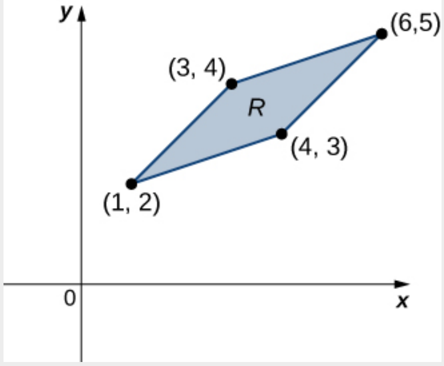
Example:

Exercise:

Problem:

Changing Variables

Consider the integral $\iint_R (x-y)dy\,dx$, where R is the parallelogram joining the points $(1,2)$, $(3,4)$, $(4,3)$, and $(6,5)$ ([link]). Make appropriate changes of variables, and write the resulting integral.



The region of integration for the given integral.

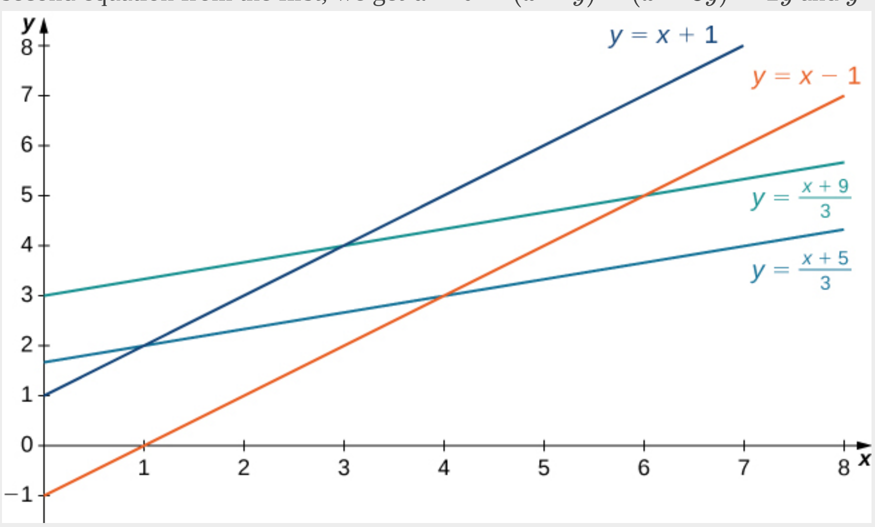
Solution:

First, we need to understand the region over which we are to integrate. The sides of the parallelogram are $x-y+1=0$, $x-y-1=0$, $x-3y+5=0$, and $x-3y+9=0$ ([link]). Another way to look at them is $x-y=-1$, $x-y=1$, $x-3y=-5$, and $x-3y=9$.

Clearly the parallelogram is bounded by the lines $y=x+1$, $y=x-1$, $y=\frac{1}{3}(x+5)$, and $y=\frac{1}{3}(x+9)$.

Notice that if we were to make $u=x-y$ and $v=x-3y$, then the limits on the integral would be $-1\leq u\leq 1$ and $-9\leq v\leq -5$.

To solve for x and y, we multiply the first equation by 3 and subtract the second equation, $3u-v=(3x-3y)-(x-3y)=2x$. Then we have $x=\frac{3u-v}{2}$. Moreover, if we simply subtract the second equation from the first, we get $u-v=(x-y)-(x-3y)=2y$ and $y=\frac{u-v}{2}$.



A parallelogram in the xy-plane that we want to transform by a change in

variables.

Thus, we can choose the transformation
Equation:

$$T(u,v)=\left(\frac{3u-v}{2},\frac{u-v}{2}\right)$$

and compute the Jacobian $J(u,v)$. We have
Equation:

$$J(u,v)=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}3/2 & -1/2\\ 1/2 & -1/2\end{vmatrix}=-\frac{3}{4}+\frac{1}{4}=-\frac{1}{2}.$$

Therefore, $|J(u,v)|=\frac{1}{2}$. Also, the original integrand becomes
Equation:

$$x-y=\frac{1}{2}[3u-v-u+v]=\frac{1}{2}[3u-u]=\frac{1}{2}[2u]=u.$$

Therefore, by the use of the transformation T, the integral changes to
Equation:

$$\iint\limits_R (x-y)dy\,dx=\int\limits_{-9}^{-5}\int\limits_{-1}^{1}J(u,v)u\,du\,dv=\int\limits_{-9}^{-5}\int\limits_{-1}^{1}\left(\frac{1}{2}\right)u\,du\,dv,$$

which is much simpler to compute.

Note:
Exercise:

Problem:

Make appropriate changes of variables in the integral $\iint\limits_R \frac{4}{(x-y)^2}dy\,dx$, where R is the trapezoid bounded by the lines $x-y=2, x-y=4, x=0,$ and $y=0$. Write the resulting integral.

Solution:

$x=\frac{1}{2}(v+u)$ and $y=\frac{1}{2}(v-u)$ and $\int\limits_{-4}^{4}\int\limits_{-2}^{2}\frac{4}{u^2}\left(\frac{1}{2}\right)du\,dv.$

Hint

Follow the steps in the previous example.

We are ready to give a problem-solving strategy for change of variables.

Note:
Problem-Solving Strategy: Change of Variables

1. Sketch the region given by the problem in the xy-plane and then write the equations of the curves that form the boundary.
2. Depending on the region or the integrand, choose the transformations $x = g(u, v)$ and $y = h(u, v)$.
3. Determine the new limits of integration in the uv-plane.
4. Find the Jacobian $J(u, v)$.
5. In the integrand, replace the variables to obtain the new integrand.
6. Replace $dy\,dx$ or $dx\,dy$, whichever occurs, by $J(u, v)du\,dv$.

In the next example, we find a substitution that makes the integrand much simpler to compute.

Example:
Exercise:

Problem:
Evaluating an Integral

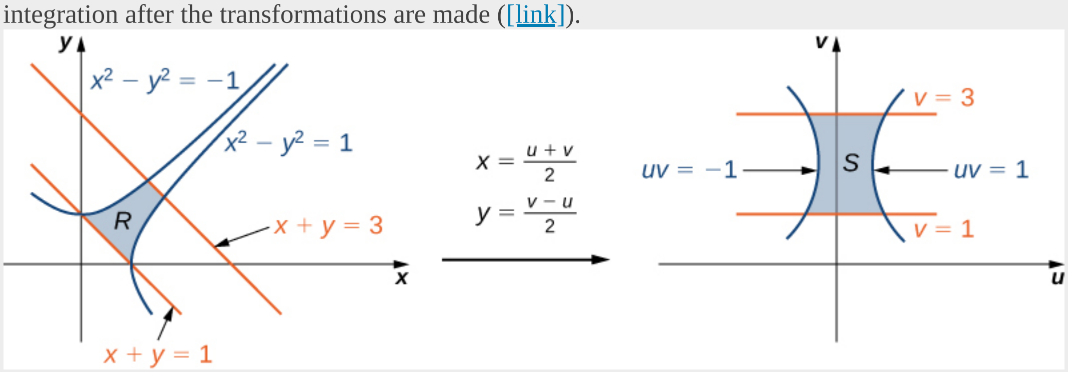
Using the change of variables $u = x - y$ and $v = x + y$, evaluate the integral
Equation:

$$\iint_R (x - y)e^{x^2 - y^2} dA,$$

where R is the region bounded by the lines $x + y = 1$ and $x + y = 3$ and the curves $x^2 - y^2 = -1$ and $x^2 - y^2 = 1$ (see the first region in [link]).

Solution:

As before, first find the region R and picture the transformation so it becomes easier to obtain the limits of integration after the transformations are made ([link]).



Transforming the region R into the region S to simplify the computation of an integral.

Given $u = x - y$ and $v = x + y$, we have $x = \frac{u+v}{2}$ and $y = \frac{v-u}{2}$ and hence the transformation to use is $T(u, v) = \left(\frac{u+v}{2}, \frac{v-u}{2}\right)$. The lines $x + y = 1$ and $x + y = 3$ become $v = 1$ and $v = 3$, respectively. The curves $x^2 - y^2 = 1$ and $x^2 - y^2 = -1$ become $uv = 1$ and $uv = -1$, respectively.

Thus we can describe the region S (see the second region [link]) as

Equation:

$$S = \left\{(u, v) | 1 \le v \le 3, \frac{-1}{v} \le u \le \frac{1}{v}\right\}.$$

The Jacobian for this transformation is

Equation:

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{vmatrix} = \frac{1}{2}.$$

Therefore, by using the transformation T, the integral changes to

Equation:

$$\iint_R (x - y) e^{x^2 - y^2} dA = \frac{1}{2} \int_1^3 \int_{-1/v}^{1/v} u e^{uv} du\, dv.$$

Doing the evaluation, we have

Equation:

$$\frac{1}{2} \int_1^3 \int_{-1/v}^{1/v} u e^{uv} du\, dv = \frac{4}{3e} \approx 0.490.$$

Note:

Exercise:

Problem:

Using the substitutions $x = v$ and $y = \sqrt{u + v}$, evaluate the integral $\iint_R y \sin(y^2 - x) dA$ where R is the region bounded by the lines $y = \sqrt{x}$, $x = 2$, and $y = 0$.

Solution:

$\frac{1}{2}(\sin 2 - 2)$

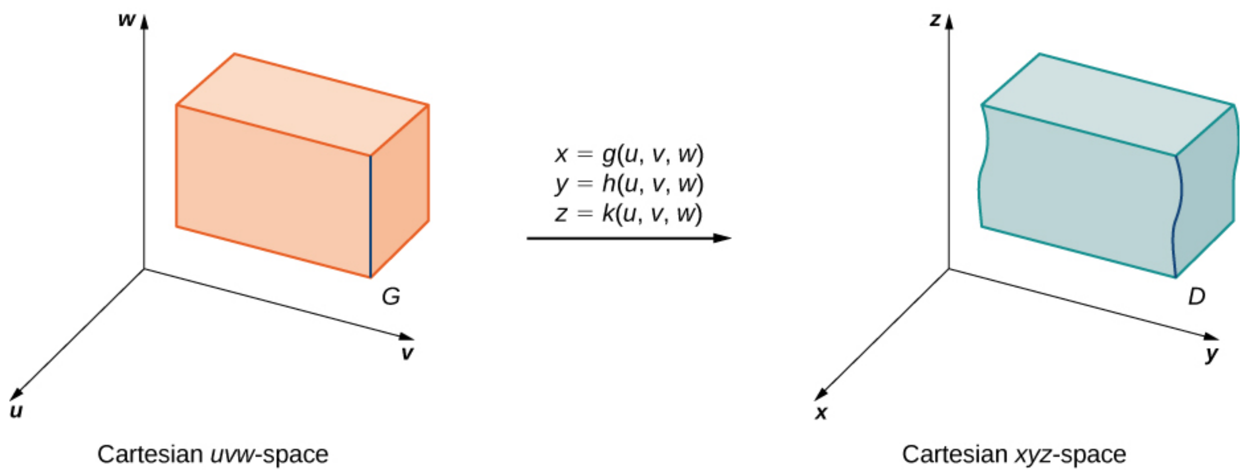
Hint

Sketch a picture and find the limits of integration.

Change of Variables for Triple Integrals

Changing variables in triple integrals works in exactly the same way. Cylindrical and spherical coordinate substitutions are special cases of this method, which we demonstrate here.

Suppose that G is a region in uvw-space and is mapped to D in xyz-space ([link]) by a one-to-one C^1 transformation $T(u, v, w) = (x, y, z)$ where $x = g(u, v, w)$, $y = h(u, v, w)$, and $z = k(u, v, w)$.



A region G in uvw-space mapped to a region D in xyz-space.

Then any function $F(x, y, z)$ defined on D can be thought of as another function $H(u, v, w)$ that is defined on G:
Equation:

$$F(x, y, z) = F(g(u, v, w), h(u, v, w), k(u, v, w)) = H(u, v, w).$$

Now we need to define the Jacobian for three variables.

Note:
Definition
The Jacobian determinant $J(u, v, w)$ in three variables is defined as follows:
Equation:

$$J(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

This is also the same as
Equation:

$$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

The Jacobian can also be simply denoted as $\frac{\partial(x,y,z)}{\partial(u,v,w)}$.

With the transformations and the Jacobian for three variables, we are ready to establish the theorem that describes change of variables for triple integrals.

Note:
Change of Variables for Triple Integrals
Let $T(u,v,w) = (x,y,z)$ where $x = g(u,v,w), y = h(u,v,w)$, and $z = k(u,v,w)$, be a one-to-one C^1 transformation, with a nonzero Jacobian, that maps the region G in the uvw-plane into the region D in the xyz-plane. As in the two-dimensional case, if F is continuous on D, then
Equation:

$$\begin{aligned} \iiint\limits_R F(x,y,z)dV &= \iiint\limits_G F(g(u,v,w), h(u,v,w), k(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\, dv\, dw \\ &= \iiint\limits_G H(u,v,w)\, |J(u,v,w)| du\, dv\, dw. \end{aligned}$$

Let us now see how changes in triple integrals for cylindrical and spherical coordinates are affected by this theorem. We expect to obtain the same formulas as in Triple Integrals in Cylindrical and Spherical Coordinates.

Example:
Exercise:

Problem:
Obtaining Formulas in Triple Integrals for Cylindrical and Spherical Coordinates

Derive the formula in triple integrals for

a. cylindrical and
b. spherical coordinates.

Solution:

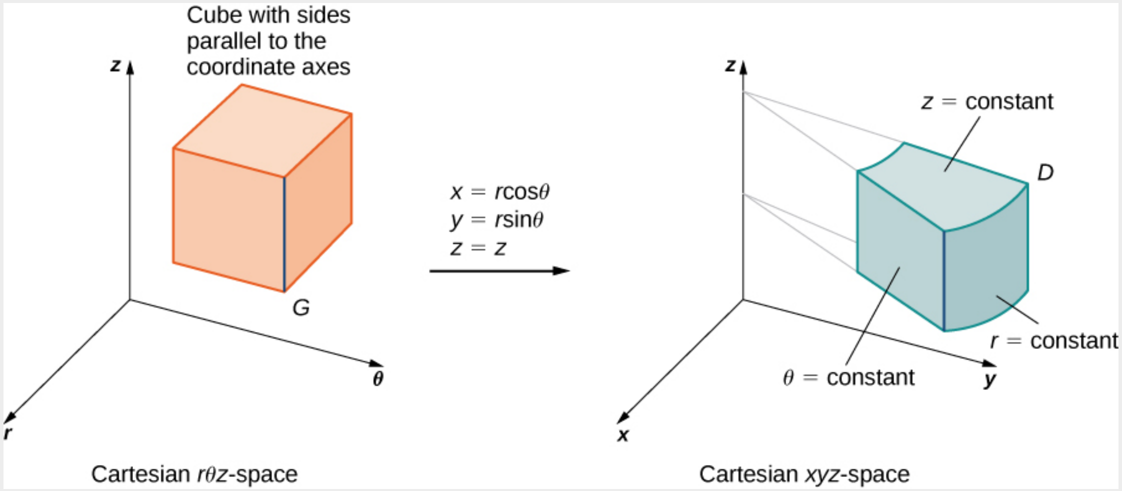
a. For cylindrical coordinates, the transformation is $T(r,\theta,z) = (x,y,z)$ from the Cartesian $r\theta z$-plane to the Cartesian xyz-plane ([link]). Here $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$. The Jacobian for the transformation is
Equation:

$$J(r,\theta,z) = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r\left(\cos^2\theta + \sin^2\theta\right) = r.$$

We know that $r \geq 0$, so $|J(r,\theta,z)| = r$. Then the triple integral is
Equation:

$$\iiint\limits_D f(x,y,z)dV = \iiint\limits_G f(r\cos\theta, r\sin\theta, z) r\,dr\,d\theta\,dz.$$



The transformation from rectangular coordinates to cylindrical coordinates can be treated as a change of variables from region G in $r\theta z$-space to region D in xyz-space.

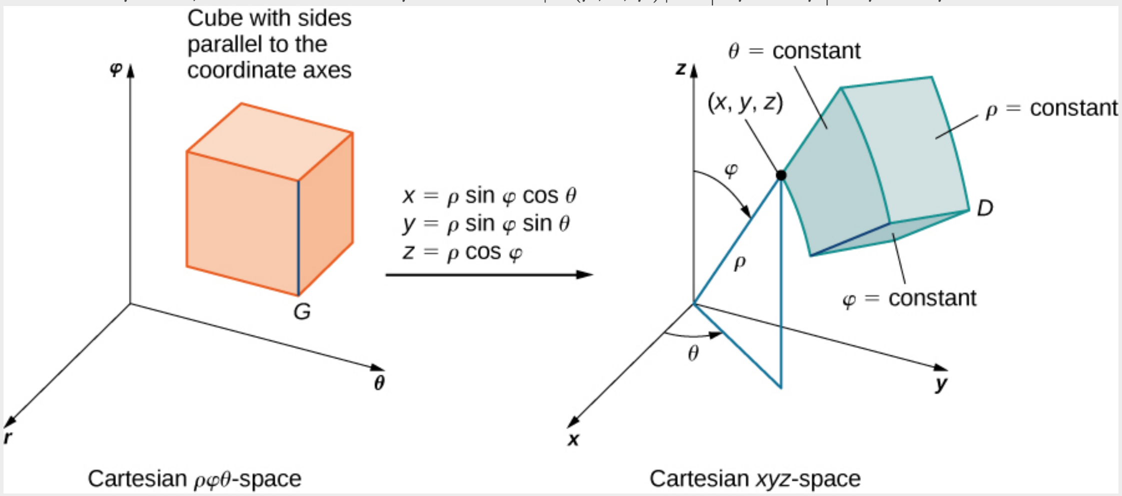
b. For spherical coordinates, the transformation is $T(\rho,\theta,\varphi) = (x,y,z)$ from the Cartesian $p\theta\varphi$-plane to the Cartesian xyz-plane ([link]). Here $x = \rho\sin\varphi\cos\theta$, $y = \rho\sin\varphi\sin\theta$, and $z = \rho\cos\varphi$. The Jacobian for the transformation is
Equation:

$$J(\rho,\theta,\varphi) = \frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & -\rho\sin\varphi\cos\theta & \rho\cos\varphi\sin\theta \\ \cos\theta & 0 & -\rho\sin\varphi \end{vmatrix}.$$

Expanding the determinant with respect to the third row:
Equation:

$$\begin{aligned}&= \cos\varphi \begin{vmatrix} -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \rho\sin\varphi\sin\theta & \rho\cos\varphi\sin\theta \end{vmatrix} - \rho\sin\varphi \begin{vmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta \end{vmatrix} \\ &= \cos\varphi\left(-\rho^2\sin\varphi\cos\varphi\sin^2\theta - \rho^2\sin\varphi\cos\varphi\cos^2\theta\right) \\ &\quad -\rho\sin\varphi\left(\rho\sin^2\varphi\cos^2\theta + \rho\sin^2\varphi\sin^2\theta\right) \\ &= -\rho^2\sin\varphi\cos^2\varphi\left(\sin^2\theta+\cos^2\theta\right) - \rho^2\sin\varphi\sin^2\varphi\left(\sin^2\theta+\cos^2\theta\right) \\ &= -\rho^2\sin\varphi\cos^2\varphi - \rho^2\sin\varphi\sin^2\varphi \\ &= -\rho^2\sin\varphi\left(\cos^2\varphi+\sin^2\varphi\right) = -\rho^2\sin\varphi.\end{aligned}$$

Since $0 \le \varphi \le \pi$, we must have $\sin\varphi \ge 0$. Thus $|J(\rho,\theta,\varphi)| = \left|-\rho^2\sin\varphi\right| = \rho^2\sin\varphi$.



The transformation from rectangular coordinates to spherical coordinates can be treated as a change of variables from region G in $\rho\theta\varphi$-space to region D in xyz-space.

Then the triple integral becomes

Equation:

$$\iiint_D f(x,y,z)dV = \iiint_G f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta.$$

Let’s try another example with a different substitution.

Example:

Exercise:

Problem:

Evaluating a Triple Integral with a Change of Variables

Evaluate the triple integral

Equation:

$$\int_0^3 \int_0^4 \int_{y/2}^{(y/2)+1} \left(x + \frac{z}{3}\right) dx\, dy\, dz$$

in xyz-space by using the transformation

Equation:

$$u = (2x - y)/2, v = y/2, \text{and } w = z/3.$$

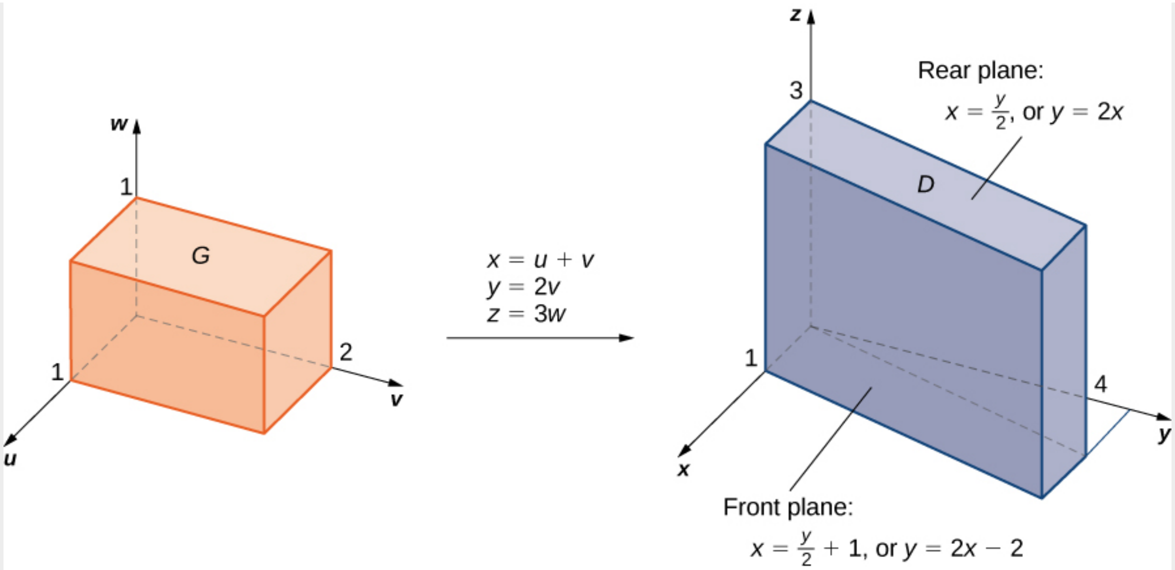
Then integrate over an appropriate region in uvw-space.

Solution:

As before, some kind of sketch of the region G in xyz-space over which we have to perform the integration can help identify the region D in uvw-space ([link]). Clearly G in xyz-space is bounded by the planes $x = y/2, x = (y/2) + 1, y = 0, y = 4, z = 0,$ and $z = 4$. We also know that we have to use $u = (2x - y)/2, v = y/2,$ and $w = z/3$ for the transformations. We need to solve for $x, y,$ and z. Here we find that $x = u + v, y = 2v,$ and $z = 3w$.

Using elementary algebra, we can find the corresponding surfaces for the region G and the limits of integration in uvw-space. It is convenient to list these equations in a table.

Equations in xyz for the region D	**Corresponding equations in uvw for the region G**	**Limits for the integration in uvw**
$x = y/2$	$u + v = 2v/2 = v$	$u = 0$
$x = y/2$	$u + v = (2v/2) + 1 = v + 1$	$u = 1$
$y = 0$	$2v = 0$	$v = 0$
$y = 4$	$2v = 4$	$v = 2$
$z = 0$	$3w = 0$	$w = 0$
$z = 3$	$3w = 3$	$w = 1$



The region G in uvw-space is transformed to region D in xyz-space.

Now we can calculate the Jacobian for the transformation:

Equation:

$$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 6.$$

The function to be integrated becomes

Equation:

$$f\left(x,y,z\right) = x + \frac{z}{3} = u + v + \frac{3w}{3} = u + v + w.$$

We are now ready to put everything together and complete the problem.

Equation:

$$\int_0^3 \int_0^4 \int_{y/2}^{(y/2)+1} \left(x + \frac{z}{3}\right) dx\, dy\, dz$$

$$= \int_0^1 \int_0^2 \int_0^1 (u+v+w)\,|J(u,v,w)|du\, dv\, dw = \int_0^1 \int_0^2 \int_0^1 (u+v+w)\,|6|du\, dv\, dw$$

$$= 6\int_0^1 \int_0^2 \int_0^1 (u+v+w)du\, dv\, dw = 6\int_0^1 \int_0^2 \left[\frac{u^2}{2} + vu + wu\right]_0^1 dv\, dw$$

$$= 6\int_0^1 \int_0^2 \left(\frac{1}{2} + v + w\right) dv\, dw = 6\int_0^1 \left[\frac{1}{2}v + \frac{v^2}{2} + wv\right]_0^2 dw$$

$$= 6\int_0^1 (3+2w)dw = 6\left[3w + w^2\right]_0^1 = 24.$$

Note:
Exercise:

Problem: Let D be the region in xyz-space defined by $1 \le x \le 2, 0 \le xy \le 2,$ and $0 \le z \le 1.$

Evaluate $\iiint_D (x^2y + 3xyz)dx\, dy\, dz$ by using the transformation $u = x, v = xy$, and $w = 3z$.

Solution:

$$\int_0^3 \int_0^2 \int_1^2 \left(\frac{v}{3} + \frac{vw}{3u}\right) du\, dv\, dw = 2 + \ln 8$$

Hint

Make a table for each surface of the regions and decide on the limits, as shown in the example.

Key Concepts

- A transformation T is a function that transforms a region G in one plane (space) into a region R in another plane (space) by a change of variables.
- A transformation $T : G \to R$ defined as $T(u,v) = (x,y)$ (or $T(u,v,w) = (x,y,z)$) is said to be a one-to-one transformation if no two points map to the same image point.
- If f is continuous on R, then $\iint_R f(x,y)dA = \iint_S f(g(u,v), h(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv.$
- If F is continuous on R, then
  Equation:

$$\begin{aligned}\iiint_R F(x,y,z)dV &= \iiint_G F(g(u,v,w),h(u,v,w),k(u,v,w))\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|du\,dv\,dw \\ &= \iiint_G H(u,v,w)\,|J(u,v,w)|du\,dv\,dw.\end{aligned}$$

In the following exercises, the function $T: S \to R, T(u,v) = (x,y)$ on the region $S = \{(u,v)|0 \le u \le 1, 0 \le v \le 1\}$ bounded by the unit square is given, where $R \subset \mathrm{R}^2$ is the image of S under T.

a. Justify that the function T is a C^1 transformation.
b. Find the images of the vertices of the unit square S through the function T.
c. Determine the image R of the unit square S and graph it.

Exercise:

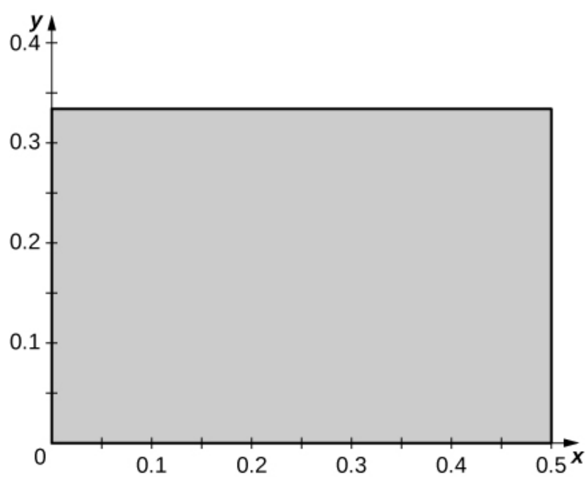
Problem: $x = 2u, y = 3v$

Exercise:

Problem: $x = \frac{u}{2}, y = \frac{v}{3}$

Solution:

a. $T(u,v) = (g(u,v), h(u,v)), x = g(u,v) = \frac{u}{2}$ and $y = h(u,v) = \frac{v}{3}$. The functions g and h are continuous and differentiable, and the partial derivatives $g_u(u,v) = \frac{1}{2}$, $g_v(u,v) = 0, h_u(u,v) = 0$ and $h_v(u,v) = \frac{1}{3}$ are continuous on S; b. $T(0,0) = (0,0)$, $T(1,0) = \left(\frac{1}{2}, 0\right), T(0,1) = \left(0, \frac{1}{3}\right)$, and $T(1,1) = \left(\frac{1}{2}, \frac{1}{3}\right)$; c. R is the rectangle of vertices $(0,0)$, $\left(\frac{1}{2}, 0\right)$, $\left(\frac{1}{2}, \frac{1}{3}\right)$, and $\left(0, \frac{1}{3}\right)$ in the xy-plane; the following figure.



Exercise:

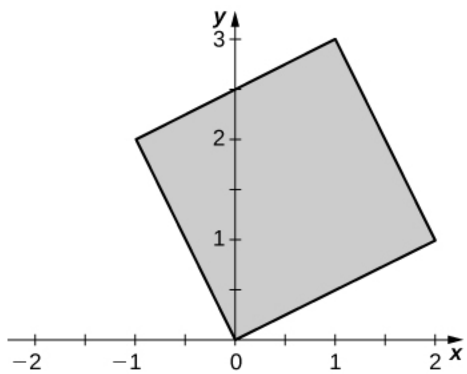
Problem: $x = u - v, y = u + v$

Exercise:

Problem: $x = 2u - v, y = u + 2v$

Solution:

a. $T(u, v) = (g(u, v), h(u, v)), x = g(u, v) = 2u - v$, and $y = h(u, v) = u + 2v$. The functions g and h are continuous and differentiable, and the partial derivatives $g_u(u, v) = 2$, $g_v(u, v) = -1$, $h_u(u, v) = 1$, and $h_v(u, v) = 2$ are continuous on S; b. $T(0, 0) = (0, 0)$, $T(1, 0) = (2, 1)$, $T(0, 1) = (-1, 2)$, and $T(1, 1) = (1, 3)$; c. R is the parallelogram of vertices $(0, 0), (2, 1), (1, 3), \text{and } (-1, 2)$ in the xy-plane; see the following figure.



Exercise:

Problem: $x = u^2, y = v^2$

Exercise:

Problem: $x = u^3, y = v^3$

Solution:

a. $T(u, v) = (g(u, v), h(u, v)), x = g(u, v) = u^3$, and $y = h(u, v) = v^3$. The functions g and h are continuous and differentiable, and the partial derivatives $g_u(u, v) = 3u^2$, $g_v(u, v) = 0$, $h_u(u, v) = 0$, and $h_v(u, v) = 3v^2$ are continuous on S; b. $T(0, 0) = (0, 0)$, $T(1, 0) = (1, 0)$, $T(0, 1) = (0, 1)$, and $T(1, 1) = (1, 1)$; c. R is the unit square in the xy-plane; see the figure in the answer to the previous exercise.

In the following exercises, determine whether the transformations $T : S \to R$ are one-to-one or not.

Exercise:

Problem: $x = u^2, y = v^2$, where S is the rectangle of vertices $(-1, 0), (1, 0), (1, 1), \text{and } (-1, 1)$.

Exercise:

Problem: $x = u^4, y = u^2 + v$, where S is the triangle of vertices $(-2, 0), (2, 0), \text{and } (0, 2)$.

Solution:

T is not one-to-one: two points of S have the same image. Indeed, $T(-2, 0) = T(2, 0) = (16, 4)$.

Exercise:

Problem: $x = 2u, y = 3v$, where S is the square of vertices $(-1, 1), (-1, -1), (1, -1), \text{and } (1, 1)$.

Exercise:

Problem: $T(u, v) = (2u - v, u)$, where S is the triangle of vertices $(-1, 1), (-1, -1)$, and $(1, -1)$.

Solution:

T is one-to-one: We argue by contradiction. $T(u_1, v_1) = T(u_2, v_2)$ implies $2u_1 - v_1 = 2u_2 - v_2$ and $u_1 = u_2$. Thus, $u_1 = u_2$ and $v_1 = v_2$.

Exercise:

Problem: $x = u + v + w, y = u + v, z = w$, where $S = R = \mathrm{R}^3$.

Exercise:

Problem: $x = u^2 + v + w, y = u^2 + v, z = w$, where $S = R = \mathrm{R}^3$.

Solution:

T is not one-to-one: $T(1, v, w) = (-1, v, w)$

In the following exercises, the transformations $T : S \to R$ are one-to-one. Find their related inverse transformations $T^{-1} : R \to S$.

Exercise:

Problem: $x = 4u, y = 5v$, where $S = R = \mathrm{R}^2$.

Exercise:

Problem: $x = u + 2v, y = -u + v$, where $S = R = \mathrm{R}^2$.

Solution:

$u = \frac{x-2y}{3}, v = \frac{x+y}{3}$

Exercise:

Problem: $x = e^{2u+v}, y = e^{u-v}$, where $S = \mathrm{R}^2$ and $R = \{(x, y) | x > 0, y > 0\}$

Exercise:

Problem: $x = \ln u, y = \ln(uv)$, where $S = \{(u, v) | u > 0, v > 0\}$ and $R = \mathrm{R}^2$.

Solution:

$u = e^x, v = e^{-x+y}$

Exercise:

Problem: $x = u + v + w, y = 3v, z = 2w$, where $S = R = \mathrm{R}^3$.

Exercise:

Problem: $x = u + v, y = v + w, z = u + w$, where $S = R = \mathrm{R}^3$.

Solution:

$u = \frac{x-y+z}{2}, v = \frac{x+y-z}{2}, w = \frac{-x+y+z}{2}$

In the following exercises, the transformation $T : S \to R, T(u, v) = (x, y)$ and the region $R \subset \mathrm{R}^2$ are given. Find the region $S \subset \mathrm{R}^2$.

Exercise:

Problem: $x = au, y = bv, R = \left\{(x, y) | x^2 + y^2 \leq a^2b^2\right\}$, where $a, b > 0$

Exercise:

Problem: $x = au, y = bv, R = \left\{(x, y) | \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\right\}$, where $a, b > 0$

Solution:

$S = \left\{(u, v) | u^2 + v^2 \leq 1\right\}$

Exercise:

Problem: $x = \frac{u}{a}, y = \frac{v}{b}, z = \frac{w}{c}, R = \left\{(x, y) | x^2 + y^2 + z^2 \leq 1\right\}$, where $a, b, c > 0$

Exercise:

Problem: $x = au, y = bv, z = cw, R = \left\{(x, y) | \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} \leq 1, z > 0\right\}$, where $a, b, c > 0$

Solution:

$R = \left\{(u, v, w) | u^2 - v^2 - w^2 \leq 1, w > 0\right\}$

In the following exercises, find the Jacobian J of the transformation.

Exercise:

Problem: $x = u + 2v, y = -u + v$

Exercise:

Problem: $x = \frac{u^3}{2}, y = \frac{v}{u^2}$

Solution:

$\frac{3}{2}$

Exercise:

Problem: $x = e^{2u-v}, y = e^{u+v}$

Exercise:

Problem: $x = ue^v, y = e^{-v}$

Solution:

-1

Exercise:

Problem: $x = u\cos(e^v), y = u\sin(e^v)$

Exercise:

Problem: $x = v\sin(u^2), y = v\cos(u^2)$

Solution:

$2uv$

Exercise:

Problem: $x = u\cosh v, y = u\sinh v, z = w$

Exercise:

Problem: $x = v\cosh\left(\frac{1}{u}\right), y = v\sinh\left(\frac{1}{u}\right), z = u + w^2$

Solution:

$\frac{v}{u^2}$

Exercise:

Problem: $x = u + v, y = v + w, z = u$

Exercise:

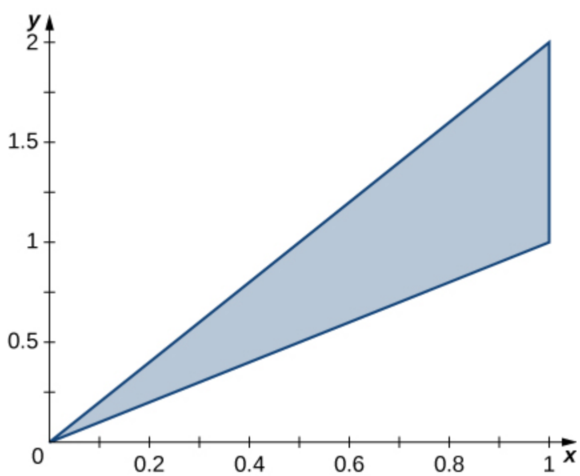
Problem: $x = u - v, y = u + v, z = u + v + w$

Solution:

2

Exercise:

Problem: The triangular region R with the vertices $(0,0)$, $(1,1)$, and $(1,2)$ is shown in the following figure.

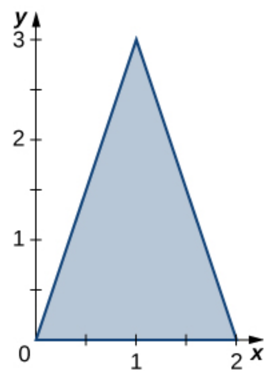


a. Find a transformation $T : S \to R$, $T(u,v) = (x,y) = (au + bv, cu + dv)$, where a, b, c, and d are real numbers with $ad - bc \neq 0$ such that $T^{-1}(0,0) = (0,0)$, $T^{-1}(1,1) = (1,0)$, and $T^{-1}(1,2) = (0,1)$.

b. Use the transformation T to find the area $A(R)$ of the region R.

Exercise:

Problem: The triangular region R with the vertices $(0,0)$, $(2,0)$, and $(1,3)$ is shown in the following figure.



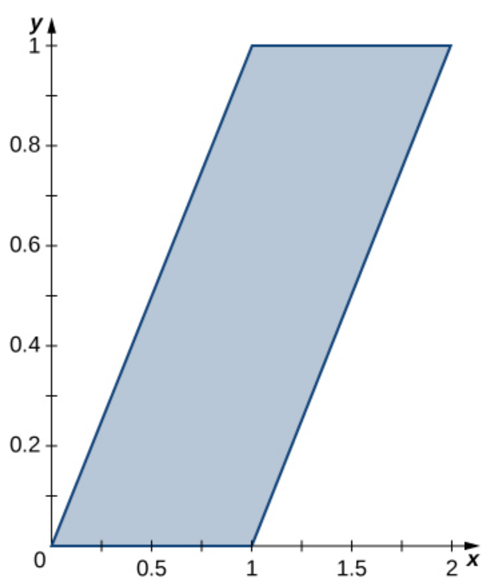
a. Find a transformation $T : S \to R$, $T(u,v) = (x,y) = (au+bv, cu+dv)$, where a, b, c and d are real numbers with $ad - bc \neq 0$ such that $T^{-1}(0,0) = (0,0)$, $T^{-1}(2,0) = (1,0)$, and $T^{-1}(1,3) = (0,1)$.
b. Use the transformation T to find the area $A(R)$ of the region R.

Solution:

a. $T(u,v) = (2u+v, 3v)$; b. The area of R is

$$A(R) = \int_0^3 \int_{y/3}^{(6-y)/3} dx\, dy = \int_0^1 \int_0^{1-u} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dv\, du = \int_0^1 \int_0^{1-u} 6 dv\, du = 3.$$

In the following exercises, use the transformation $u = y - x$, $v = y$, to evaluate the integrals on the parallelogram R of vertices $(0,0)$, $(1,0)$, $(2,1)$, and $(1,1)$ shown in the following figure.



Exercise:

Problem: $\iint_R (y-x)dA$

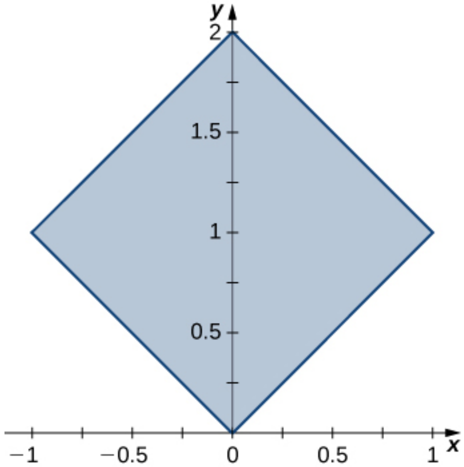
Exercise:

Problem: $\iint_R (y^2-xy)dA$

Solution:

$-\frac{1}{4}$

In the following exercises, use the transformation $y-x=u, x+y=v$ to evaluate the integrals on the square R determined by the lines $y=x, y=-x+2, y=x+2$, and $y=-x$ shown in the following figure.



Exercise:

Problem: $\iint_R e^{x+y}dA$

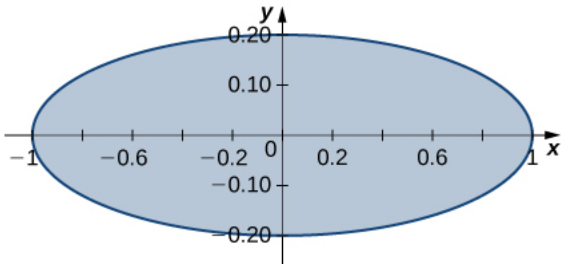
Exercise:

Problem: $\iint_R \sin(x-y)dA$

Solution:

$-1+\cos 2$

In the following exercises, use the transformation $x=u, 5y=v$ to evaluate the integrals on the region R bounded by the ellipse $x^2+25y^2=1$ shown in the following figure.



Exercise:

Problem: $\iint\limits_R \sqrt{x^2 + 25y^2}\, dA$

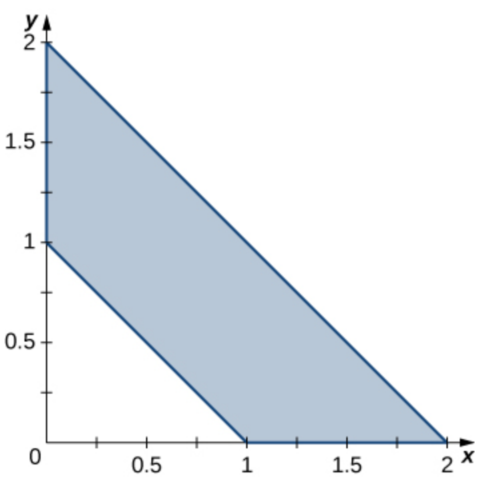
Exercise:

Problem: $\iint\limits_R \left(x^2 + 25y^2\right)^2 dA$

Solution:

$\frac{\pi}{15}$

In the following exercises, use the transformation $u = x + y, v = x - y$ to evaluate the integrals on the trapezoidal region R determined by the points $(1, 0), (2, 0), (0, 2), \text{and } (0, 1)$ shown in the following figure.



Exercise:

Problem: $\iint\limits_R \left(x^2 - 2xy + y^2\right) e^{x+y} dA$

Exercise:

Problem: $\iint\limits_R \left(x^3 + 3x^2y + 3xy^2 + y^3\right) dA$

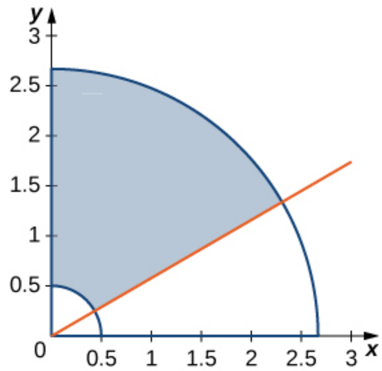
Solution:

$\frac{31}{5}$

Exercise:

Problem:

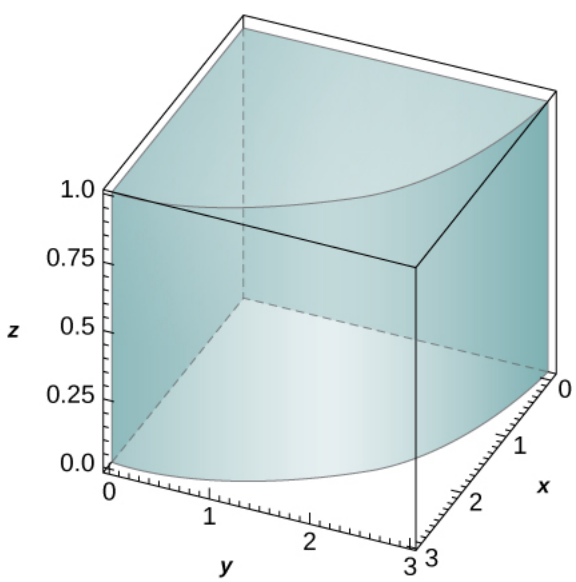
The circular annulus sector R bounded by the circles $4x^2 + 4y^2 = 1$ and $9x^2 + 9y^2 = 64$, the line $x = y\sqrt{3}$, and the y-axis is shown in the following figure. Find a transformation T from a rectangular region S in the $r\theta$-plane to the region R in the xy-plane. Graph S.



Exercise:

Problem:

The solid R bounded by the circular cylinder $x^2 + y^2 = 9$ and the planes $z = 0, z = 1, x = 0,$ and $y = 0$ is shown in the following figure. Find a transformation T from a cylindrical box S in $r\theta z$-space to the solid R in xyz-space.



Solution:

$T(r, \theta, z) = (r \cos \theta, r \sin \theta, z)$; $S = [0, 3] \times \left[0, \frac{\pi}{2}\right] \times [0, 1]$ in the $r\theta z$-space

Exercise:

Problem:

Show that $\iint_R f\left(\sqrt{\frac{x^2}{3}+\frac{y^2}{3}}\right)dA = 2\pi\sqrt{15}\int_0^1 f(\rho)\rho\,d\rho$, where f is a continuous function on $[0,1]$ and R is the region bounded by the ellipse $5x^2+3y^2=15$.

Exercise:

Problem:

Show that $\iiint_R f\left(\sqrt{16x^2+4y^2+z^2}\right)dV = \frac{\pi}{2}\int_0^1 f(\rho)\rho^2 d\rho$, where f is a continuous function on $[0,1]$ and R is the region bounded by the ellipsoid $16x^2+4y^2+z^2=1$.

Exercise:

Problem:

[T] Find the area of the region bounded by the curves $xy=1, xy=3, y=2x$, and $y=3x$ by using the transformation $u=xy$ and $v=\frac{y}{x}$. Use a computer algebra system (CAS) to graph the boundary curves of the region R.

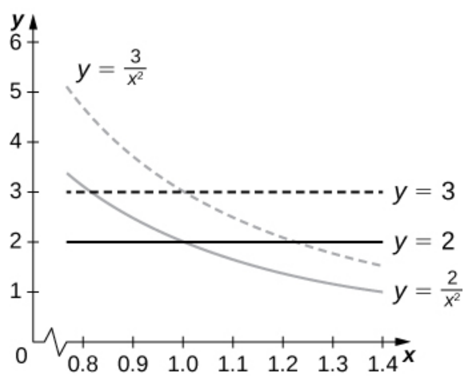
Exercise:

Problem:

[T] Find the area of the region bounded by the curves $x^2y=2, x^2y=3, y=x$, and $y=2x$ by using the transformation $u=x^2y$ and $v=\frac{y}{x}$. Use a CAS to graph the boundary curves of the region R.

Solution:

The area of R is $10-4\sqrt{6}$; the boundary curves of R are graphed in the following figure.



Exercise:

Problem:

Evaluate the triple integral $\int_0^1\int_1^2\int_z^{z+1}(y+1)dx\,dy\,dz$ by using the transformation $u=x-z$, $v=3y$, and $w=\frac{z}{2}$.

Exercise:

Problem:

Evaluate the triple integral $\int_0^2 \int_4^6 \int_{3z}^{3z+2} (5-4y)dx\,dz\,dy$ by using the transformation $u = x - 3z, v = 4y$, and $w = z$.

Solution:

8

Exercise:

Problem:

A transformation $T : \mathrm{R}^2 \to \mathrm{R}^2, T(u,v) = (x,y)$ of the form $x = au + bv, y = cu + dv$, where a, b, c, and d are real numbers, is called linear. Show that a linear transformation for which $ad - bc \neq 0$ maps parallelograms to parallelograms.

Exercise:

Problem:

The transformation $T_\theta : \mathrm{R}^2 \to \mathrm{R}^2, T_\theta(u,v) = (x,y)$, where $x = u\cos\theta - v\sin\theta$, $y = u\sin\theta + v\cos\theta$, is called a rotation of angle θ. Show that the inverse transformation of T_θ satisfies $T_\theta{}^{-1} = T_{-\theta}$, where $T_{-\theta}$ is the rotation of angle $-\theta$.

Exercise:

Problem:

[T] Find the region S in the uv-plane whose image through a rotation of angle $\frac{\pi}{4}$ is the region R enclosed by the ellipse $x^2 + 4y^2 = 1$. Use a CAS to answer the following questions.

a. Graph the region S.
b. Evaluate the integral $\iint_S e^{-2uv} du\,dv$. Round your answer to two decimal places.

Exercise:

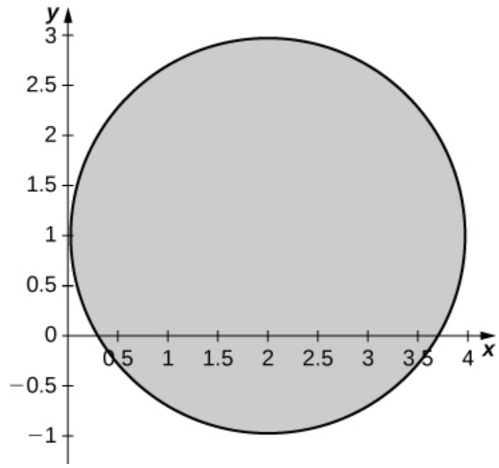
Problem:

[T] The transformations $T_i : \mathbb{R}^2 \to \mathbb{R}^2$, $i = 1, \ldots, 4$, defined by $T_1(u,v) = (u,-v)$, $T_2(u,v) = (-u,v)$, $T_3(u,v) = (-u,-v)$, and $T_4(u,v) = (v,u)$ are called reflections about the x-axis, y-axis, origin, and the line $y = x$, respectively.

a. Find the image of the region $S = \left\{(u,v) | u^2 + v^2 - 2u - 4v + 1 \leq 0\right\}$ in the xy-plane through the transformation $T_1 \circ T_2 \circ T_3 \circ T_4$.
b. Use a CAS to graph R.
c. Evaluate the integral $\iint_S \sin(u^2) du\,dv$ by using a CAS. Round your answer to two decimal places.

Solution:

a. $R = \left\{(x,y) | y^2 + x^2 - 2y - 4x + 1 \leq 0\right\}$; b. R is graphed in the following figure;



c. 3.16

Exercise:

Problem:

[T] The transformation $T_{k,1,1} : \mathbb{R}^3 \to \mathbb{R}^3, T_{k,1,1}(u, v, w) = (x, y, z)$ of the form $x = ku, y = v, z = w,$ where $k \neq 1$ is a positive real number, is called a stretch if $k > 1$ and a compression if $0 < k < 1$ in the x-direction. Use a CAS to evaluate the integral $\iiint\limits_S e^{-(4x^2+9y^2+25z^2)}\,dx\,dy\,dz$ on the solid $S = \{(x, y, z) | 4x^2 + 9y^2 + 25z^2 \leq 1\}$ by considering the compression $T_{2,3,5}(u, v, w) = (x, y, z)$ defined by $x = \frac{u}{2}, y = \frac{v}{3}$, and $z = \frac{w}{5}$. Round your answer to four decimal places.

Exercise:

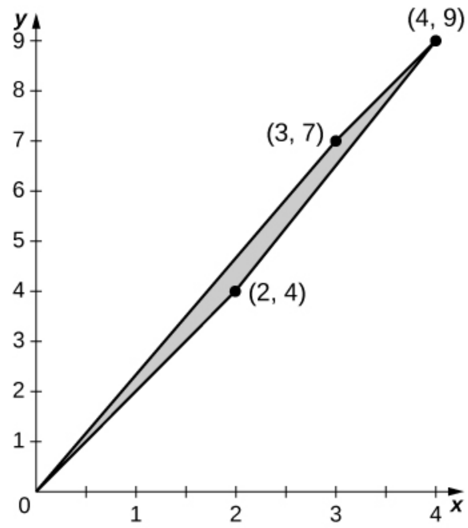
Problem:

[T] The transformation $T_{a,0} : \mathbb{R}^2 \to \mathbb{R}^2, T_{a,0}(u, v) = (u + av, v)$, where $a \neq 0$ is a real number, is called a shear in the x-direction. The transformation, $T_{b,0} : \mathrm{R}^2 \to \mathrm{R}^2, T_{o,b}(u, v) = (u, bu + v)$, where $b \neq 0$ is a real number, is called a shear in the y-direction.

a. Find transformations $T_{0,2} \circ T_{3,0}$.
b. Find the image R of the trapezoidal region S bounded by $u = 0, v = 0, v = 1$, and $v = 2 - u$ through the transformation $T_{0,2} \circ T_{3,0}$.
c. Use a CAS to graph the image R in the xy-plane.
d. Find the area of the region R by using the area of region S.

Solution:

a. $T_{0,2} \circ T_{3,0}(u, v) = (u + 3v, 2u + 7v)$; b. The image S is the quadrilateral of vertices $(0, 0), (3, 7), (2, 4), \text{and } (4, 9)$; c. S is graphed in the following figure;



d. $\frac{3}{2}$

Exercise:

Problem:

Use the transformation, $x = au, y = av, z = cw$ and spherical coordinates to show that the volume of a region bounded by the spheroid $\frac{x^2+y^2}{a^2} + \frac{z^2}{c^2} = 1$ is $\frac{4\pi a^2 c}{3}$.

Exercise:

Problem:

Find the volume of a football whose shape is a spheroid $\frac{x^2+y^2}{a^2} + \frac{z^2}{c^2} = 1$ whose length from tip to tip is 11 inches and circumference at the center is 22 inches. Round your answer to two decimal places.

Solution:

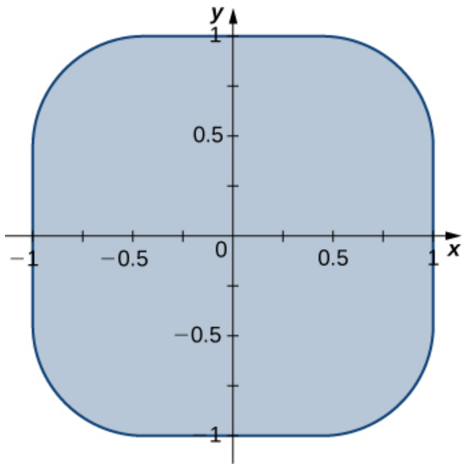
$\frac{2662}{3\pi} \simeq 282.45 \text{ in}^3$

Exercise:

Problem:

[T] Lamé ovals (or superellipses) are plane curves of equations $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 1$, where *a*, *b*, and *n* are positive real numbers.

a. Use a CAS to graph the regions R bounded by Lamé ovals for $a = 1, b = 2, n = 4$ and $n = 6$, respectively.
b. Find the transformations that map the region R bounded by the Lamé oval $x^4 + y^4 = 1$, also called a squircle and graphed in the following figure, into the unit disk.



c. Use a CAS to find an approximation of the area $A(R)$ of the region R bounded by $x^4 + y^4 = 1$. Round your answer to two decimal places.

Exercise:

Problem:

[T] Lamé ovals have been consistently used by designers and architects. For instance, Gerald Robinson, a Canadian architect, has designed a parking garage in a shopping center in Peterborough, Ontario, in the shape of a superellipse of the equation $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 1$ with $\frac{a}{b} = \frac{9}{7}$ and $n = e$. Use a CAS to find an approximation of the area of the parking garage in the case $a = 900$ yards, $b = 700$ yards, and $n = 2.72$ yards.

Solution:

$A(R) \simeq$ 83,999.2

Chapter Review Exercises

True or False? Justify your answer with a proof or a counterexample.

Exercise:

Problem: $\int_a^b \int_c^d f(x, y) dy\, dx = \int_c^d \int_a^b f(x, y) dy\, dx$

Exercise:

Problem: Fubini's theorem can be extended to three dimensions, as long as f is continuous in all variables.

Solution:

True.

Exercise:

Problem: The integral $\int_0^{2\pi} \int_0^1 \int_r^1 dz\, dr\, d\theta$ represents the volume of a right cone.

Exercise:

Problem: The Jacobian of the transformation for $x = u^2 - 2v, y = 3v - 2uv$ is given by $-4u^2 + 6u + 4v$.

Solution:

False.

Evaluate the following integrals.

Exercise:

Problem: $\iint\limits_R \left(5x^3y^2 - y^2\right) dA, R = \{(x, y)|0 \le x \le 2, 1 \le y \le 4\}$

Exercise:

Problem: $\iint\limits_D \frac{y}{3x^2 + 1} dA, D = \{(x, y)|0 \le x \le 1, -x \le y \le x\}$

Solution:

0

Exercise:

Problem: $\iint\limits_D \sin\left(x^2 + y^2\right) dA$ where D is a disk of radius 2 centered at the origin

Exercise:

Problem: $\int\limits_0^1 \int\limits_y^1 xye^{x^2} dx\, dy$

Solution:

$\frac{1}{4}$

Exercise:

Problem: $\int\limits_{-1}^1 \int\limits_0^z \int\limits_0^{x-z} 6dy\, dx\, dz$

Exercise:

Problem: $\iiint\limits_R 3y\, dV$, where $R = \left\{(x, y, z)|0 \le x \le 1, 0 \le y \le x, 0 \le z \le \sqrt{9 - y^2}\right\}$

Solution:

1.475

Exercise:

Problem: $\int_0^2 \int_0^{2\pi} \int_r^1 r\, dz\, d\theta\, dr$

Exercise:

Problem: $\int_0^{2\pi} \int_0^{\pi/2} \int_1^3 \rho^2 \sin(\varphi) d\rho\, d\varphi\, d\theta$

Solution:

$\frac{52}{3}\pi$

Exercise:

Problem: $\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz\, dy\, dx$

For the following problems, find the specified area or volume.

Exercise:

Problem: The area of region enclosed by one petal of $r = \cos(4\theta)$.

Solution:

$\frac{\pi}{16}$

Exercise:

Problem: The volume of the solid that lies between the paraboloid $z = 2x^2 + 2y^2$ and the plane $z = 8$.

Exercise:

Problem: The volume of the solid bounded by the cylinder $x^2 + y^2 = 16$ and from $z = 1$ to $z + x = 2$.

Solution:

93.291

Exercise:

Problem:

The volume of the intersection between two spheres of radius 1, the top whose center is $(0, 0, 0.25)$ and the bottom, which is centered at $(0, 0, 0)$.

For the following problems, find the center of mass of the region.

Exercise:

Problem: $\rho(x, y) = xy$ on the circle with radius 1 in the first quadrant only.

Solution:

$\left(\frac{8}{15}, \frac{8}{15}\right)$

Exercise:

Problem: $\rho(x, y) = (y + 1)\sqrt{x}$ in the region bounded by $y = e^x$, $y = 0$, and $x = 1$.

Exercise:

Problem: $\rho(x, y, z) = z$ on the inverted cone with radius 2 and height 2.

Solution:

$\left(0, 0, \frac{8}{5}\right)$

Exercise:

Problem:

The volume an ice cream cone that is given by the solid above $z = \sqrt{(x^2 + y^2)}$ and below $z^2 + x^2 + y^2 = z$.

The following problems examine Mount Holly in the state of Michigan. Mount Holly is a landfill that was converted into a ski resort. The shape of Mount Holly can be approximated by a right circular cone of height 1100 ft and radius 6000 ft.

Exercise:

Problem:

If the compacted trash used to build Mount Holly on average has a density $400\,\text{lb/ft}^3$, find the amount of work required to build the mountain.

Solution:

$1.452\pi \times 10^{15}$ ft-lb

Exercise:

Problem:

In reality, it is very likely that the trash at the bottom of Mount Holly has become more compacted with all the weight of the above trash. Consider a density function with respect to height: the density at the top of the mountain is still density $400\,\text{lb/ft}^3$ and the density increases. Every 100 feet deeper, the density doubles. What is the total weight of Mount Holly?

The following problems consider the temperature and density of Earth’s layers.

Exercise:

Problem:

[T] The temperature of Earth’s layers is exhibited in the table below. Use your calculator to fit a polynomial of degree 3 to the temperature along the radius of the Earth. Then find the average temperature of Earth. (*Hint*: begin at 0 in the inner core and increase outward toward the surface)

Layer	Depth from center (km)	Temperature $^{\circ}C$
Rocky Crust	0 to 40	0
Upper Mantle	40 to 150	870
Mantle	400 to 650	870
Inner Mantel	650 to 2700	870
Molten Outer Core	2890 to 5150	4300
Inner Core	5150 to 6378	7200

Source: http://www.enchantedlearning.com/subjects/astronomy/planets/earth/Inside.shtml

Solution:

$y = -1.238 \times 10^{-7}x^3 + 0.001196x^2 - 3.666x + 7208$; average temperature approximately $2800^{\circ}C$

Exercise:

Problem:

[T] The density of Earth's layers is displayed in the table below. Using your calculator or a computer program, find the best-fit quadratic equation to the density. Using this equation, find the total mass of Earth.

Layer	Depth from center (km)	Density (g/cm3)
Inner Core	0	12.95
Outer Core	1228	11.05
Mantle	3488	5.00
Upper Mantle	6338	3.90
Crust	6378	2.55

Source: http://hyperphysics.phy-astr.gsu.edu/hbase/geophys/earthstruct.html

The following problems concern the Theorem of Pappus (see Moments and Centers of Mass for a refresher), a method for calculating volume using centroids. Assuming a region R, when you revolve around the x-axis the volume is given by $V_x = 2\pi A\bar{y}$, and when you revolve around the y-axis the volume is given by $V_y = 2\pi A\bar{x}$, where A is the area of R. Consider the region bounded by $x^2 + y^2 = 1$ and above $y = x + 1$.

Exercise:

Problem: Find the volume when you revolve the region around the x-axis.

Solution:

$\frac{\pi}{3}$

Exercise:

Problem: Find the volume when you revolve the region around the y-axis.

Glossary

Jacobian

the Jacobian $J\left(u, v\right)$ in two variables is a 2×2 determinant:

Equation:

$$J\left(u, v\right) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix};$$

the Jacobian $J\left(u, v, w\right)$ in three variables is a 3×3 determinant:

Equation:

$$J(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}$$

one-to-one transformation

a transformation $T : G \rightarrow R$ defined as $T\left(u, v\right) = \left(x, y\right)$ is said to be one-to-one if no two points map to the same image point

planar transformation

a function T that transforms a region G in one plane into a region R in another plane by a change of variables

transformation

a function that transforms a region G in one plane into a region R in another plane by a change of variables